Signals as elements of a vector space

For interest only.

Abstract vector spaces

A vector space V is a set that is closed under finite vector addition and scalar multiplication. The following must hold for any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and scalars a, b (real or complex):

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (commutativity)

2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity)

3. There exists an element $\mathbf{0}$ in V such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$ (additive identity)

4. For all $\mathbf{v}$ in V there exists an element $\mathbf{w}$ in V such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$ (existence of additive inverse)

5. $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$ (distributivity of vector sums)

6. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ (distributivity of scalar sums)

7. $a(b\mathbf{v}) = (ab)\mathbf{v}$ (associativity of scalar multiplication)

8. $1\mathbf{v} = \mathbf{v}$ (scalar multiplication identity).

Euclidean space

Euclidean n-space $V = \mathbb{R}^n$ is a familiar example of a vector space, where every element is represented by a list of n real numbers, scalars are real numbers, addition is componentwise, and scalar multiplication is multiplication on each term separately.

Vector addition:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

Scalar multiplication:

$$a \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} av_1 \\ \vdots \\ av_n \end{pmatrix}$$

Exercise: Convince yourself that the 8 vector space axioms hold for $V = \mathbb{R}^3$.

Canonical basis of Euclidean space

Given a vector $\mathbf{x} \in \mathbb{R}^n$ (n-dimensional Euclidean space) we can write $\mathbf{x}$ as a linear combination (weighted sum) of n vectors:

$$\mathbf{x} = a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} a_i \mathbf{e}_i$$

where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)^T$ with the single nonzero element at position i. Clearly in this case $a_i = x_i$.

The set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ (or alternatively $\{\mathbf{e}_i\}_{i=1}^n$) forms a basis of $\mathbb{R}^n$: any values for the weights $\{a_i\}_{i=1}^n$ determines a particular $\mathbf{x}$, and any $\mathbf{x}$ is uniquely specified by a set of coefficient values $\{a_i\}_{i=1}^n$.

This can be written in the form $\mathbf{x} = \mathbf{I}\mathbf{a}$, where $\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 \cdots \mathbf{e}_n \end{pmatrix}$ represents the canonical basis and $\mathbf{a} = \mathbf{x}$ are the coefficients.

So...

Suppose I'm thinking of a vector $\mathbf{x}$ in $\mathbb{R}^3$ and you want to know what vector it is.

You could get some information by asking me "What is the value of the first component relative to the canonical basis?", in which case I might reply "The value is 11." If you asked me the value for the other two components you would know the overall vector.

Alternatively you could ask me "What value do you obtain when you take the dot product of your vector with $\mathbf{e}_1$, in which case I would also reply "The value is 11". If I additionally gave you the dot products with vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ you would know $\{a_1, a_2, a_3\}$ and $\mathbf{x}$ is fully determined via $\mathbf{x} = \sum_{i=1}^n a_i \mathbf{e}_i$.

General basis of Euclidean space

For $\mathbb{R}^n$, any set of linearly independent vectors $\{\mathbf{b}_i\}_{i=1}^n$ is a basis. Thus for any $\mathbf{x} \in \mathbb{R}^n$ we can write

$$\mathbf{x} = a_1' \mathbf{b}_1 + a_2' \mathbf{b}_2 + \cdots + a_n' \mathbf{b}_n = \sum_{i=1}^n a_i' \mathbf{b}_i = \mathbf{B}\mathbf{a}',$$

where $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 \cdots \mathbf{b}_n \end{pmatrix}$. Since

$$\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{I}\mathbf{B}\mathbf{B}^{-1}\mathbf{x} = \mathbf{B}(\mathbf{B}^{-1}\mathbf{x})$$

the coefficients of $\mathbf{x}$ on the basis $\mathbf{B}$ are evidently $\mathbf{a}' = \mathbf{B}^{-1}\mathbf{x}$.

One often chooses the basis vectors to be orthogonal and unit length, so $\mathbf{b}_i^T \mathbf{b}_j = \delta_{ij} \implies \mathbf{B}^T \mathbf{B} = \mathbf{I} \implies \mathbf{B}^{-1} = \mathbf{B}^T$. In this case $\mathbf{a}' = \mathbf{B}^T \mathbf{x}$, so the ith element of $\mathbf{a}'$ is $a_i' = \mathbf{b}_i^T \mathbf{x}$.

The vector space of continuous functions

Let C be the set of real-valued continuous functions of a real variable. Elements of this space are *functions*, written for example as $s(t)$.

For elements (functions) $u(t)$ and $v(t)$ in C we can define a vector addition operation, which takes the two elements and returns a third: $u(t) + v(t)$: this is a function which, for each t, has a value equal to the sum of the values of the two functions at the same value of t.

For $v(t)$ in C and a scalar a we can define scalar multiplication in the obvious way, and the product of the scalar with the function is a new function $av(t)$: this is a function which, for each t, has a value equal to a times the value of the function at t.

The vector space axioms hold on the set C with the above operations. Therefore C is a vector space, and the elements can be considered to be vectors.

Exercise: Convince yourself that the 8 vector space axioms hold for C defined above.

Why do we care?

Consider say a temperature sensor on a buoy floating on the Atlantic Ocean. For each instant in time, this sensor provides a temperature reading. Thus for each time instant t, the temperature signal takes on a value $s(t)$. It is therefore natural to represent signals by functions.

If $s(t)$ is a continuous function, then it is a member of C (the vector space of continuous functions). We can therefore think of the signal $s(t)$ as a *vector* of this space.

This is probably the best way to think of signals — as vectors.

There are other properties of signals that make them special, but when working with them you should keep the analogy of signals as (generalised) vectors in mind.

Basis of continuous function space

Since C is a vector space, it has a basis. Any element $x(t)$ of this space can therefore be written as a linear combination (weighted sum) of a set of basis vectors $b_i(t)$. The only complication is that C is infinite dimensional, so there are infinitely many basis vectors!

If the set of basis vectors is $\{b_i(t)\}_{i=-\infty}^{\infty}$, then any $x(t)$ in C can be written as

$$x(t) = \sum_{i=-\infty}^{\infty} a_i b_i(t)$$

for some appropriate choice of coefficients $\{a_i\}_{i=-\infty}^{\infty}$, where i is integer. Knowing these coefficients is *entirely equivalent* to knowing the original signal $x(t)$.

The Fourier series applies to signals $x(t)$ that are periodic with period T. In this case the set of functions $\{b_i(t)\}_{i=-\infty}^{\infty}$ with $b_i(t) = e^{j2\pi it/T}$ and i integer is a basis. This is just the set of complex exponentials with frequencies $2\pi i/T$ for appropriate values of i. The Fourier series decomposition therefore lets us write any periodic signal $x(t)$ as a linear combination or weighted sum of a set of complex exponentials. The coefficients or weights specify the amount of each complex exponential present in the signal.

Orthogonality of the Fourier series basis functions

The basis functions $b_i(t)$ have an important property — they are orthogonal over the range 0 to T. In particular, it is quite easy to show that

$$\int_0^T b_k(t)b_l^*(t)dt = \delta_{kl}T, \quad \text{where} \quad \delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l. \end{cases}$$

Exercise: Convince yourself that this is true. Using the previous definition we have

$$\int_0^T b_i(t)b_k^*(t)dt = \int_0^T e^{j2\pi it/T}e^{-j2\pi kt/T}dt = \int_0^T e^{j2\pi(i-k)t/T}dt = \int_0^T e^{j2\pi mt/T}dt,$$

where $m = i - k$ is an integer. This integral is trivially equal to T when $m = 0$. Show that for nonzero m the integral evaluates to $\frac{1}{j2\pi m/T}(e^{j2\pi m} - 1) = 0$, since $e^{j2\pi m} = (e^{j2\pi})^m = 1^m = 1$. Additionally, we could change the integration limits to go from a to $T + a$ and the result would still be true.

Finding the coefficients

Multiplying both sides of $x(t) = \sum_{i=-\infty}^{\infty} a_i b_i(t)$ by $b_k^*(t)$ and integrating:

$$\int_0^T x(t)b_k^*(t)dt = \int_0^T \left[\sum_{i=-\infty}^{\infty} a_i b_i(t)\right] b_k^*(t)dt = \sum_{i=-\infty}^{\infty} a_i \int_0^T b_i(t)b_k^*(t)dt$$

$$= \sum_{i=-\infty}^{\infty} a_i \delta_{ik}T = a_k T.$$

Thus for each k we have $a_k = \frac{1}{T} \int_0^T x(t) e^{-j2\pi kt/T} dt$.

So...

Suppose I'm thinking of a signal $s(t)$ and you want to know what signal it is.

You could get some information by asking me "What is the value of the signal at $t = 1$?", in which case I might reply "The value of $s(1)$ is 14." If you asked me the value for *lots* of different values of t (infinitely many), you would know the overall signal. Thus the signal could be described by the infinite set of values $\{\ldots, x(0), \ldots, x(0.5), \ldots, x(1), \ldots\}$.

Alternatively you could also get some information by asking me "If you take your signal $s(t)$ and calculate $a_i = \frac{1}{T} \int_0^T x(t) e^{-j2\pi it/T} dt$ for $k = 1$ what do you get?", I might reply "You get a_1 is equal to 2.". If you asked me this question for every k you would know the signal, since you could calculate its value at any instant in time using $x(t) = \sum_{i=-\infty}^{\infty} a_i e^{j2\pi it/T}$. Thus the signal could also be described by the infinite set of values $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$.

It turns out that the second description is much more useful in practice.

	Euclidean space $\mathbb{R}^n$	Function space C (periodic T)
Dimensionality of space	n	∞
Vectors in space	$\mathbf{x}$	$x(t)$
Inner (dot) product	$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$	$\langle x(t), y(t) \rangle = \int_0^T x^*(t) y(t) dt$
Orthonormal basis vectors	$\mathbf{b}_1, \ldots, \mathbf{b}_n$	$\ldots, b_{-1}(t), b_0(t), b_1(t), \ldots$
	with $\langle \mathbf{b}_k, \mathbf{b}_l \rangle = \delta_{kl}$	with $\langle b_k(t), b_l(t) \rangle = \delta_{kl}$
Representation on basis	$\mathbf{x} = \sum_{i=1}^n c_i \mathbf{b}_i$ for some $\{c_i\}_{i=1}^n$	$x(t) = \sum_{i=-\infty}^{\infty} c_i b_i(t)$ for some $\{c_i\}_{i=-\infty}^{\infty}$
Analysis	$c_i = \langle \mathbf{b}_i, \mathbf{x} \rangle = \mathbf{b}_i^T \mathbf{x}$	$c_i = \langle b_i(t), x(t) \rangle = \int_0^T x(t) b_i^*(t) dt$

Exercise: Suppose $b_i(t) = \frac{1}{T} e^{j2\pi it/T}$. Convince yourself that the relationships in the function space $\mathcal{C}$ above correspond exactly to the conventional Fourier series relationships.