Tutorial: Systems in time and frequency domains

Answers to these questions will *not* be made available.

The italicised blocks contain hints and guidelines on how to approach the questions, and would not appear in an exam. If you understand the concepts required in these questions, then in my opinion you understand the content of the course. Note that it might take a few hours to work through these questions properly.

---

1. A system has an impulse response

$$h(t) = e^{-4(t-1)}u(t-1).$$

Find the response of the system to the input

$$x(t) = e^{-2t}u(t)$$

using

(a) Time domain convolution

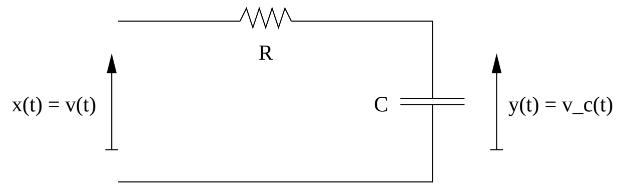
Straightforward, and you know how to do this: the output is $y(t) = h(t) * x(t)$, which you find by integration. I find it easier to shift $h(t)$ backwards by one so that it starts at the origin, do the convolution, and shift the answer forwards by one to get the required answer. This uses the property that $y(t+1) = h(t+1) * x(t)$.

(b) Frequency domain multiplication.

Convolution in the time domain corresponds to multiplication in the frequency domain: $Y(\omega) = H(\omega)X(\omega)$. Find Fourier transforms, do the multiplication, and then find the inverse transform. You'll probably need a partial fraction expansion to find this inverse from tables. Your answer should be the same as in the previous case.

---

2. The following circuit



is governed by the differential equation

$$C\frac{dv_c(t)}{dt} = \frac{v(t) - v_c(t)}{R}$$

(by now you should be able to confirm this). Suppose that for some application we want to take $x(t) = v(t)$ as input and $y(t) = v_c(t)$ as output, then the input-output relation is

$$C\frac{dy(t)}{dt} = \frac{x(t) - y(t)}{R}.$$

(a) Find the frequency response of the system and plot its magnitude and phase.

You need to know that the frequency response is given by $H(\omega) = Y(\omega)/X(\omega)$, and can be found by taking Fourier transforms of the input-output relation above and manipulating. The plots of magnitude ($|H(\omega)|$, or sometimes $20\log_{10}|H(\omega)|$) and phase ($\angle H(\omega)$) are called the **Bode plot** *of the system. For this question you should see that the system passes low frequencies, but that the magnitude response falls off at high frequencies — this is a simple* **lowpass filter**.

(b) Find the impulse response of the system.

The frequency response of the system (also called the transfer function of the system) is given by the Fourier transform of the impulse response, so $H(\omega) = \mathcal{F}\{h(t)\}$. Thus the impulse response can be found by taking the inverse Fourier transform of the impulse response: $h(t) = \mathcal{F}^{-1}\{H(\omega)\}$.

Important extension: repeat the question, but swap the resistor and the capacitor in the circuit and take the output to be the voltage across the resistor $y(t) = v_r(t)$ instead. You should find that the system obeys the equation

$$C\left(\frac{dx(t)}{dt} - \frac{dy(t)}{dt}\right) = \frac{1}{R}y(t),$$

and the Bode plot will show you that the system is a **highpass** *filter.*

---

3. A LTI system has an impulse response

$$h(t) = e^{-4(t-1)}u(t-1).$$

Use two different methods to find the response of the system to the input

$$x(t) = \sin(4\pi t).$$

Method 1: for a LTI system the response to a complex exponential input $x_c(t) = e^{j\omega_0 t}$ *is* $y_c(t) = H(\omega_0)e^{j\omega_0 t}$ *(you should be able to prove this using convolution), where* $H(\omega)$ *is the frequency response of the system (the Fourier transform of the impulse response* $h(t)$*). By writing* $x(t) = \sin(4\pi t)$ *in terms of complex exponentials you can use this to find the output. Note that it is possible to massage the result so that the output signal is a real-valued sinusoid at frequency* $\omega_0 = 4\pi$ *(with some scaling and shift which you can determine).*

Method 2: The action of the system can be represented as $Y(\omega) = H(\omega)X(\omega)$ *in the frequency domain. Take the Fourier transforms of* $h(t)$ *and* $x(t)$*, and multiply them to get* $Y(\omega)$*. This should consist of two Dirac delta functions with different sizes, one at* $\omega = -4\pi$ *and one at* $\omega = 4\pi$*. You will need to use the sifting property to find the sizes of these delta functions:* $H(\omega)\delta(\omega - \omega_0) = H(\omega_0)\delta(\omega - \omega_0)$*. Finding the inverse Fourier transform of* $Y(\omega)$ *gives the required* $y(t)$*. Your result should be the same as for the first method.*
