Fourier series and an RC circuit

Circuit

Resistor R in series with capacitor C, input $x(t)$ is voltage across combination, output $y(t)$ is voltage across capacitor.

Resistor:

$$v_r(t) = Ri(t)$$

Capacitor:

$$i(t) = C\frac{d}{dt}v_c(t)$$

and

$$v(t) = v_c(t) + v_r(t)$$

Eliminating $i(t)$ and letting $v(t) = x(t)$ and $y(t) = v_c(t)$ gives DE

$$x(t) = RC\frac{dy(t)}{dt} + y(t)$$

System interpretation

The system is linear (linear constant coefficient DE), and therefore has an impulse response $h(t)$. (Don't yet know how to find it.)

The input/output relationship can therefore be written in the form

$$y(t) = h(t) * x(t)$$

Input complex exponential $x(t) = c_1 e^{j\omega_0 t}$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda = \int_{-\infty}^{\infty} h(\lambda) c_1 e^{j\omega_0 (t-\lambda)} d\lambda \\ &= \left(c_1 \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega_0 \lambda} d\lambda \right) e^{j\omega_0 t} = d_1 e^{j\omega_0 t} \end{aligned}$$

Finding output coefficients

Return to DE:

$$x(t) = RC\frac{dy(t)}{dt} + y(t)$$

Consider input $x(t) = c_1 e^{j\omega_0 t}$. Know the output is of the form $y(t) = d_1 e^{j\omega_0 t}$. Coefficient c_1 is known: want to find d_1.

Substitute into DE and solve:

$$\begin{aligned} c_1 e^{j\omega_0 t} &= RC\frac{d}{dt}\left[d_1 e^{j\omega_0 t}\right] + d_1 e^{j\omega_0 t} \\ \implies c_1 e^{j\omega_0 t} &= j\omega_0 RC d_1 e^{j\omega_0 t} + d_1 e^{j\omega_0 t} \\ \implies c_1 &= (j\omega_0 RC + 1) d_1 \end{aligned}$$

So

$$d_1 = \frac{1}{1 + j\omega_0 RC} c_1$$

Multiple component signal

Input complex exponential $x(t) = c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t}$:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda \\
&= \int_{-\infty}^{\infty} h(\lambda) c_1 e^{j\omega_0 (t-\lambda)} d\lambda + \int_{-\infty}^{\infty} h(\lambda) c_2 e^{j2\omega_0 (t-\lambda)} d\lambda \\
&= \left(c_1 \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega_0 \lambda} d\lambda \right) e^{j\omega_0 t} \\
&\qquad + \left(c_2 \int_{-\infty}^{\infty} h(\lambda) e^{-j2\omega_0 \lambda} d\lambda \right) e^{j2\omega_0 t} \\
&= d_1 e^{j\omega_0 t} + d_2 e^{j2\omega_0 t}
\end{aligned}
$$

Complex exponentials in, complex exponentials out at same frequencies.
Only need to find the coefficients d_1 and d_2 given c_1 and c_2

Coefficients for multi-component signal

You can (*and should*) do same for two-component signal. Let

$$x(t) = c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t},$$

$y(t)$ must be of form

$$y(t) = d_1 e^{j\omega_0 t} + d_2 e^{j2\omega_0 t},$$

substitute into the DE and solve (algebraically!) for d_1 and d_2.

In general, if input and output are

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{and} \qquad y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t},$$

then

$$d_k = \frac{1}{1 + jk\omega_0 RC} c_k$$

In general

If there is a LCCDE linking inputs and outputs, the FS coefficients for input and output will be found to obey

$$d_k = H(k\omega_0)c_k$$

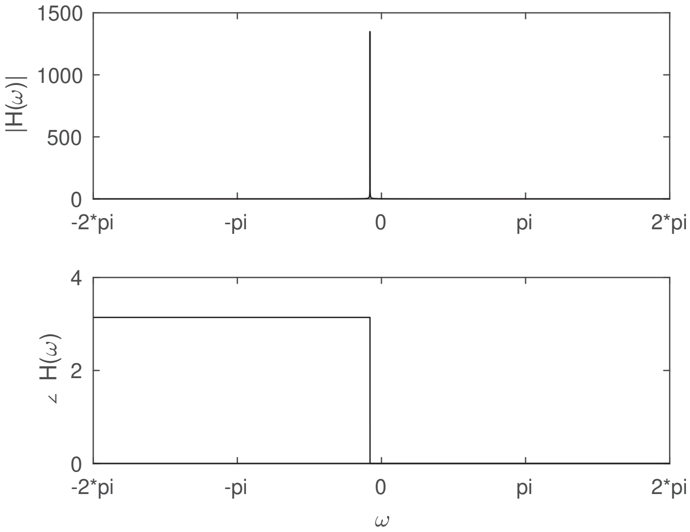
where for the RC circuit we have

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

To work out how a signal is modified by the circuit, only need to know the values of $H(\omega)$ at the frequencies present

$H(\omega)$ is just a complex number for each value of frequency ω. The plot of the magnitude and phase of H as a function of ω is called the *Bode plot* of the system.

Bode plot for RC circuit



Observe

Time domain description of signals $x(t)$ and $y(t)$ $\longrightarrow$ *differential equation* linking input and output of system. Can solve for $y(t)$ given $x(t)$, but no intuition.

Instead, think about signals as (weighted linear) combinations of complex exponentials (or combinations of frequencies)

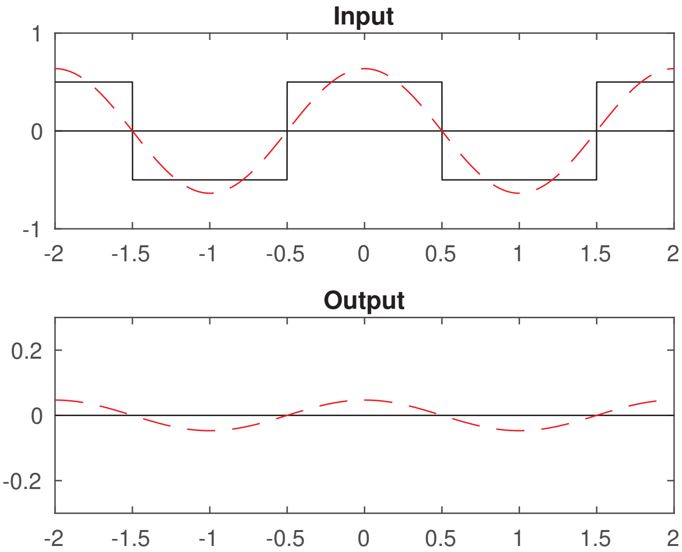
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{and} \qquad y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

$\longrightarrow$ *algebraic equation* linking input and output:

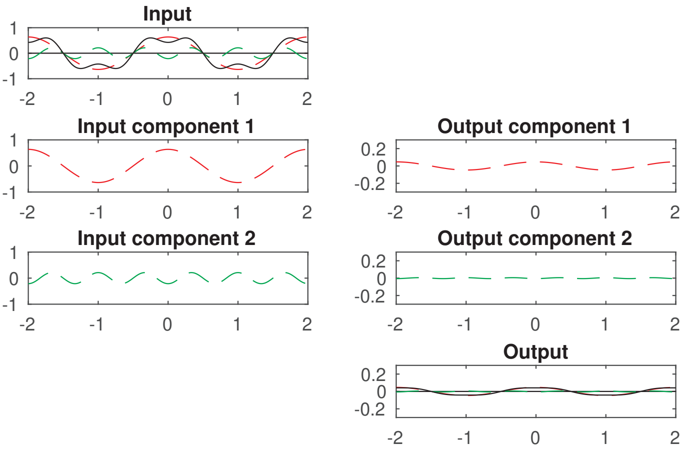
$$d_k = H(k\omega_0)c_k$$

Much easier to understand.

One component approximation to square wave



Two component approximation to square wave



Many component approximation to square wave