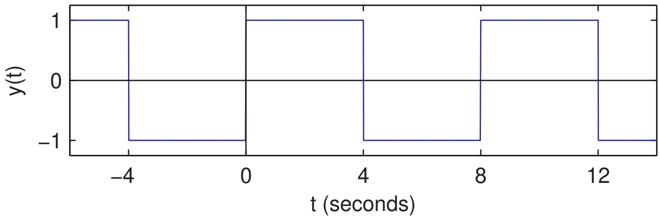
Fourier series: Additional notes

Linking Fourier series representations for signals

Rectangular waveform

Require FS expansion of signal $y(t)$ below:



Period $T = 8$, so $\omega_0 = 2\pi/T = 2\pi/8 = \pi/4$ and the signal has a FS representation

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t}.$$

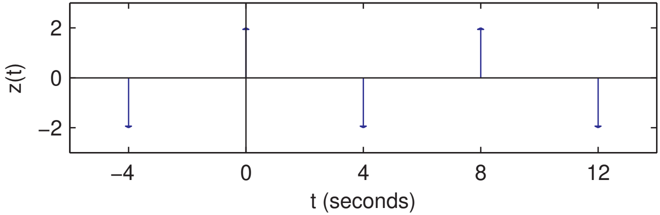
We could find the FS coefficients using the formula

$$d_k = \frac{1}{8}\int_0^8 y(t)e^{-jk(\pi/4)t}dt = \frac{1}{8}\int_0^4 (1)e^{-jk(\pi/4)t}dt + \frac{1}{8}\int_4^8 (-1)e^{-jk(\pi/4)t}dt$$

but will do something simpler (and more interesting) instead.

Rectangular waveform: derivative signal

Consider instead the derivative of the previous signal $z(t) = \frac{d}{dt}y(t)$:



This also has a period $T = 8$, and a FS representation

$$z(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk(\pi/4)t}.$$

Note that this is of the same form as the FS for $y(t)$, just with a different set of coefficients.

FS coefficients for derivative signal

Find the FS coefficients for $z(t)$ by integrating over one complete period. Choose the interval $t = -2$ to $t = 6$ to avoid integrating over half a Dirac delta (we're free to choose the integration range as long as we integrate over one period):

$$\begin{aligned}
f_k &= \frac{1}{8}\int_{-2}^{6} z(t)e^{-jk(\pi/4)t}dt = \frac{1}{8}\int_{-2}^{6}[2\delta(t) - 2\delta(t-4)]e^{-jk(\pi/4)t}dt \\
&= \frac{2}{8}\int_{-2}^{6}\delta(t)e^{-jk(\pi/4)t}dt - \frac{2}{8}\int_{-2}^{6}\delta(t-4)e^{-jk(\pi/4)t}dt \\
&= \frac{1}{4}\int_{-2}^{6}\delta(t)e^{-jk(\pi/4)0}dt - \frac{1}{4}\int_{-2}^{6}\delta(t-4)e^{-jk(\pi/4)4}dt \\
&= \frac{1}{4}e^{-jk(\pi/4)0}\int_{-2}^{6}\delta(t)dt - \frac{1}{4}e^{-jk(\pi/4)4}\int_{-2}^{6}\delta(t-4)dt \\
&= \frac{1}{4}e^{-jk(\pi/4)0} - \frac{1}{4}e^{-jk(\pi/4)4} \\
&= \frac{1}{4}(1 - e^{-jk\pi}).
\end{aligned}$$

Rectangular waveform: Relationship between coefficients

Since $z(t) = \frac{d}{dt}y(t)$, we can find a relationship between the coefficients:

$$z(t) = \frac{d}{dt}y(t) = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} \frac{d}{dt}\left[d_k e^{jk(\pi/4)t}\right]$$

$$= \sum_{k=-\infty}^{\infty} d_k \frac{d}{dt}\left[e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} d_k jk(\pi/4) e^{jk(\pi/4)t}.$$

But the FS for $y(t)$ is of the same form

$$z(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk(\pi/4)t},$$

so each of the coefficients must be equal:

$$f_k = d_k jk(\pi/4) = jk\frac{\pi}{4}d_k.$$

Rectangular waveform: Relationship between coefficients

The FS coefficients d_k (for the square wave) are related to the coefficients f_k (for the derivative signal) by

$$d_k = \frac{1}{jk\pi/4} f_k.$$

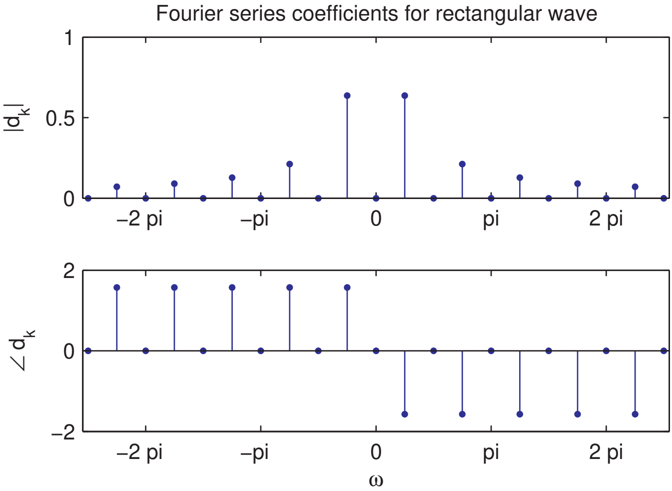
This formula will work for all k except $k = 0$ (where it becomes an indeterminate form — division by zero). To find d_0, just go back to $y(t)$ and calculate directly:

$$d_0 = \int_0^8 y(t)dt = \frac{1}{8}\int_0^4 (1)dt + \frac{1}{8}\int_4^8 (-1)dt = 0.$$

Therefore the FS coefficients d_k of $y(t)$ are

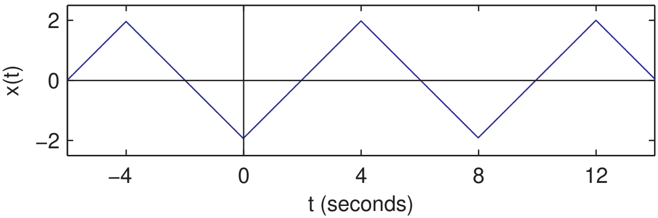
$$d_k = \begin{cases} 0 & (k = 0) \\ \frac{1}{jk\pi}(1 - e^{-jk\pi}) & (k \neq 0). \end{cases}$$

Rectangular wave: FS coefficient plot



Triangular waveform

Now find the FS coefficients of $x(t)$ below:



Period $T = 8$, so $\omega_0 = 2\pi/T = 2\pi/8 = \pi/4$ and the signal has a FS representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(\pi/4)t}.$$

Triangular waveform: FS coefficients

We could find the FS coefficients using the formula

$$
\begin{aligned}
c_k &= \frac{1}{8}\int_0^8 x(t)e^{-jk(\pi/4)t}dt \\
&= \frac{1}{8}\int_0^4 x(t)e^{-jk(\pi/4)t}dt + \frac{1}{8}\int_4^8 x(t)e^{-jk(\pi/4)t}dt \\
&= \frac{1}{8}\int_0^4 (t-2)e^{-jk(\pi/4)t}dt + \frac{1}{8}\int_4^8 (4-t)e^{-jk(\pi/4)t}dt
\end{aligned}
$$

so

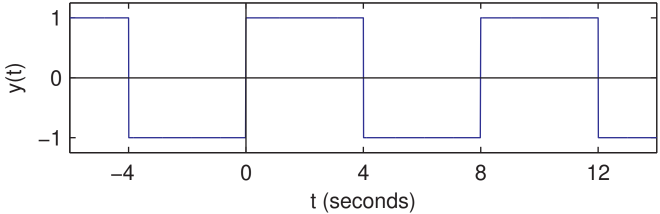
$$
\begin{aligned}
c_k = \frac{1}{8}\int_0^4 te^{-jk(\pi/4)t} - \frac{2}{8}\int_0^4 e^{-jk(\pi/4)t} \\
+ \frac{4}{8}\int_4^8 e^{-jk(\pi/4)t} - \frac{1}{8}\int_4^8 te^{-jk(\pi/4)t}.
\end{aligned}
$$

Two of these integrals are easy, but two have to be done by parts.

Exercise 1: Calculate the FS coefficients for the above.

Triangular waveform: alternative method

Consider instead the signal $y(t) = \frac{d}{dt}x(t)$:



This also has a period $T = 8$, and a FS representation

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t}$$

and the coefficients were found earlier to be

$$d_k = \begin{cases} 0 & (k = 0) \\ \frac{1}{jk\pi}(1 - e^{-jk\pi}) & (k \neq 0). \end{cases}$$

Triangular waveform: Relationship between coefficients

As before, since $y(t) = \frac{d}{dt}x(t)$ a relationship exists between the coefficients:

$$y(t) = \frac{d}{dt}x(t) = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} c_k e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} \frac{d}{dt}\left[c_k e^{jk(\pi/4)t}\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{d}{dt}\left[e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} c_k jk(\pi/4) e^{jk(\pi/4)t}.$$

But the FS for $y(t)$ is of the same form

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t},$$

so each of the coefficients must be equal:

$$d_k = c_k jk(\pi/4) = jk\frac{\pi}{4}c_k.$$

Triangular waveform: Relationship between coefficients

The FS coefficients c_k (for the triangular wave) can therefore be found from the coefficients d_k (for the square wave) using

$$c_k = \frac{1}{jk\pi/4} d_k,$$

as long as $k \neq 0$. To find c_0, just go back to $x(t)$ and calculate directly:

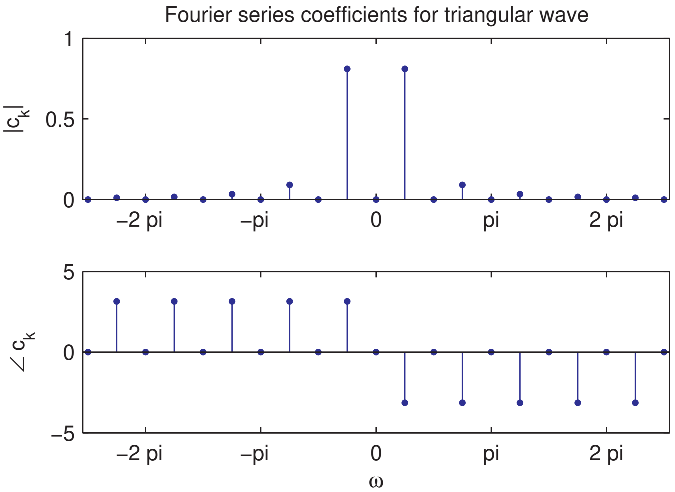
$$c_0 = \frac{1}{8}\int_0^8 x(t)dt = \frac{1}{8}\int_0^4 (t-2)dt + \frac{1}{8}\int_4^8 (4-t)dt = 0.$$

The FS coefficients for $x(t)$ are therefore

$$c_k = \begin{cases} 0 & (k=0) \\ \frac{1}{jk\pi/4}\frac{1}{jk\pi}(1-e^{-jk\pi}) & (k \neq 0). \end{cases}$$

No integration by parts needed!

Triangular wave: FS coefficient plot

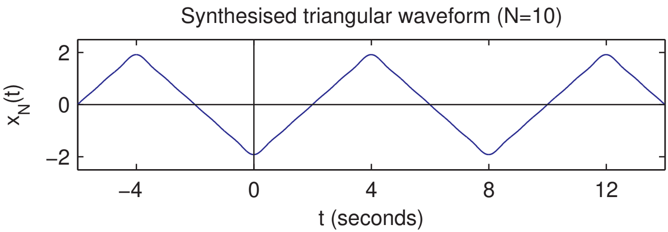


Check with Matlab

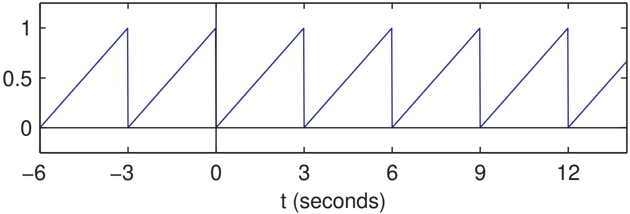
```
j = sqrt(-1);  % to be sure
tv = -6:0.001:14;  % time values (a row vector)
xv = zeros(size(tv));  % signal values initially zero
N = 10;  % highest term in synthesis equation
for k=-N:N
  % Current complex exponential values (a row vector)
  xcv = exp(j*k*pi/4*tv);

  % Coefficient for current complex exponential
  if k==0
    ck = 0;  % DC
  else
    ck = 4/((j*k*pi)^2)*(1 - exp(-j*k*pi));  % formula
    %ck = 1/(j*k*pi)*(1 - exp(-j*k*pi));  % y(t)
    %ck = 1/4*(1 - exp(-j*k*pi));  % z(t)
  end

  % Add scaled complex exponential to signal values
  xv = xv + ck*xcv;
end
plot(tv,real(xv));  % the values *should* be real
```

Exercise 2: Run the above code in Matlab.

Triangular wave: synthesis



Exercise 3: Find the FS of the signal below:



Notes

Some interesting observations:

1. Triangular waveform is continuous (quite smooth), but has discontinuous derivatives — FS coefficients decreases as $\frac{1}{\omega^2}$ — they decrease very quickly as frequency increases.
2. Rectangular waveform is discontinuous (less smooth than triangular waveform) — FS coefficients decreases as $\frac{1}{\omega}$ — more slowly.
3. Smooth waveforms generally contain less high frequency components — their coefficients go to zero closer to the origin in the frequency domain.
4. Alternatively, to reconstruct a signal with fast variations (i.e. discontinuities) requires large components at high frequencies.

Linking coefficients for other transformations

Consider a signal $x(t)$ with period T: it has a FS expansion

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with $\omega_0 = 2\pi/T$.

Let $y(t)$ be a signal related to $x(t)$. If $y(t)$ is also periodic with period T then it has a FS expansion

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}.$$

and the coefficients c_k and d_k are related.

Relationship for shift transformation

Consider the case where $y(t) = x(t - a)$ for some a. Clearly $y(t)$ is periodic with the same period as $x(t)$, and

$$y(t) = x(t-a) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-a)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jk\omega_0 a}$$
$$= \sum_{k=-\infty}^{\infty} (c_k e^{-jk\omega_0 a}) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}.$$

The coefficients are therefore related by

$$d_k = c_k e^{-jk\omega_0 a}$$

Thus a shift corresponds to a change in the phase of each coefficient, where the amount of the change is proportional to the size of the shift.

Exercise 4: Find the FS of the signal below, using the series coefficients c_k found earlier for $x(t)$: