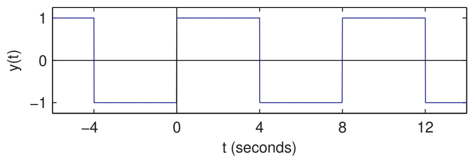
Fourier series: Additional notes

1 Linking Fourier series representations for signals

1.1 Rectangular waveform

Require FS expansion of signal $y(t)$ below:



Period $T = 8$, so $\omega_0 = 2\pi/T = 2\pi/8 = \pi/4$ and the signal has a FS representation

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t}.$$

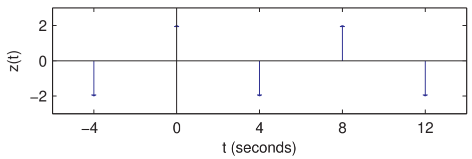
We could find the FS coefficients using the formula

$$d_k = \frac{1}{8}\int_0^8 y(t)e^{-jk(\pi/4)t}dt = \frac{1}{8}\int_0^4 (1)e^{-jk(\pi/4)t}dt + \frac{1}{8}\int_4^8 (-1)e^{-jk(\pi/4)t}dt$$

but will do something simpler (and more interesting) instead.

Rectangular waveform: derivative signal

Consider instead the derivative of the previous signal $z(t) = \frac{d}{dt}y(t)$:



This also has a period $T = 8$, and a FS representation

$$z(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk(\pi/4)t}.$$

Note that this is of the same form as the FS for $y(t)$, just with a different set of coefficients.

FS coefficients for derivative signal

Find the FS coefficients for $z(t)$ by integrating over one complete period. Choose the interval $t = -2$ to $t = 6$ to avoid integrating over half a Dirac delta (we're free to choose the integration range as long as we integrate over one period):

$$\begin{aligned}
f_k &= \frac{1}{8}\int_{-2}^{6} z(t)e^{-jk(\pi/4)t}dt = \frac{1}{8}\int_{-2}^{6}[2\delta(t) - 2\delta(t-4)]e^{-jk(\pi/4)t}dt \\
&= \frac{2}{8}\int_{-2}^{6}\delta(t)e^{-jk(\pi/4)t}dt - \frac{2}{8}\int_{-2}^{6}\delta(t-4)e^{-jk(\pi/4)t}dt \\
&= \frac{1}{4}\int_{-2}^{6}\delta(t)e^{-jk(\pi/4)0}dt - \frac{1}{4}\int_{-2}^{6}\delta(t-4)e^{-jk(\pi/4)4}dt \\
&= \frac{1}{4}e^{-jk(\pi/4)0}\int_{-2}^{6}\delta(t)dt - \frac{1}{4}e^{-jk(\pi/4)4}\int_{-2}^{6}\delta(t-4)dt \\
&= \frac{1}{4}e^{-jk(\pi/4)0} - \frac{1}{4}e^{-jk(\pi/4)4} \\
&= \frac{1}{4}(1 - e^{-jk\pi}).
\end{aligned}$$

Rectangular waveform: Relationship between coefficients

Since $z(t) = \frac{d}{dt}y(t)$, we can find a relationship between the coefficients:

$$\begin{aligned}
z(t) = \frac{d}{dt}y(t) &= \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty}\frac{d}{dt}\left[d_k e^{jk(\pi/4)t}\right] \\
&= \sum_{k=-\infty}^{\infty} d_k \frac{d}{dt}\left[e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} d_k jk(\pi/4)e^{jk(\pi/4)t}.
\end{aligned}$$

But the FS for $y(t)$ is of the same form

$$z(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk(\pi/4)t},$$

so each of the coefficients must be equal:

$$f_k = d_k jk(\pi/4) = jk\frac{\pi}{4}d_k.$$

The FS coefficients d_k (for the square wave) are related to the coefficients f_k (for the derivative signal) by

$$d_k = \frac{1}{jk\pi/4}f_k.$$

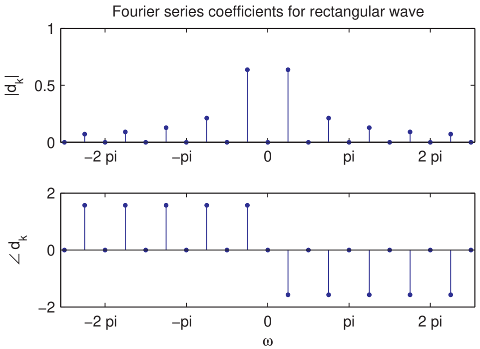
This formula will work for all k except $k = 0$ (where it becomes an indeterminate form — division by zero). To find d_0, just go back to $y(t)$ and calculate directly:

$$d_0 = \int_0^8 y(t)dt = \frac{1}{8}\int_0^4 (1)dt + \frac{1}{8}\int_4^8 (-1)dt = 0.$$

Therefore the FS coefficients d_k of $y(t)$ are

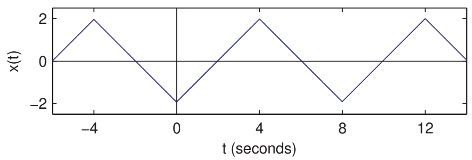
$$d_k = \begin{cases} 0 & (k = 0) \\ \frac{1}{jk\pi}(1 - e^{-jk\pi}) & (k \neq 0). \end{cases}$$

Rectangular wave: FS coefficient plot



1.2 Triangular waveform

Now find the FS coefficients of $x(t)$ below:



Period $T = 8$, so $\omega_0 = 2\pi/T = 2\pi/8 = \pi/4$ and the signal has a FS representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(\pi/4)t}.$$

Triangular waveform: FS coefficients

We could find the FS coefficients using the formula

$$\begin{aligned} c_k &= \frac{1}{8}\int_0^8 x(t)e^{-jk(\pi/4)t}dt \\ &= \frac{1}{8}\int_0^4 x(t)e^{-jk(\pi/4)t}dt + \frac{1}{8}\int_4^8 x(t)e^{-jk(\pi/4)t}dt \\ &= \frac{1}{8}\int_0^4 (t-2)e^{-jk(\pi/4)t}dt + \frac{1}{8}\int_4^8 (4-t)e^{-jk(\pi/4)t}dt \end{aligned}$$

so

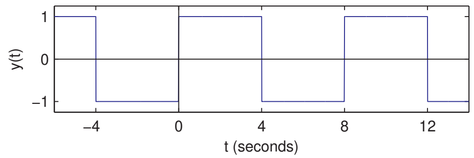
$$c_k = \frac{1}{8}\int_0^4 te^{-jk(\pi/4)t} - \frac{2}{8}\int_0^4 e^{-jk(\pi/4)t} + \frac{4}{8}\int_4^8 e^{-jk(\pi/4)t} - \frac{1}{8}\int_4^8 te^{-jk(\pi/4)t}.$$

Two of these integrals are easy, but two have to be done by parts.

Exercise 1: Calculate the FS coefficients for the above.

Triangular waveform: alternative method

Consider instead the signal $y(t) = \frac{d}{dt}x(t)$:



This also has a period $T = 8$, and a FS representation

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t}$$

and the coefficients were found earlier to be

$$d_k = \begin{cases} 0 & (k=0) \\ \frac{1}{jk\pi}(1-e^{-jk\pi}) & (k \neq 0). \end{cases}$$

Triangular waveform: Relationship between coefficients

As before, since $y(t) = \frac{d}{dt}x(t)$ a relationship exists between the coefficients:

$$\begin{aligned} y(t) = \frac{d}{dt}x(t) &= \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} c_k e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} \frac{d}{dt}\left[c_k e^{jk(\pi/4)t}\right] \\ &= \sum_{k=-\infty}^{\infty} c_k \frac{d}{dt}\left[e^{jk(\pi/4)t}\right] = \sum_{k=-\infty}^{\infty} c_k jk(\pi/4)e^{jk(\pi/4)t}. \end{aligned}$$

But the FS for $y(t)$ is of the same form

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk(\pi/4)t},$$

so each of the coefficients must be equal:

$$d_k = c_k jk(\pi/4) = jk\frac{\pi}{4}c_k.$$

The FS coefficients c_k (for the triangular wave) can therefore be found from the coefficients d_k (for the square wave) using

$$c_k = \frac{1}{jk\pi/4}d_k,$$

as long as $k \neq 0$. To find c_0, just go back to $x(t)$ and calculate directly:

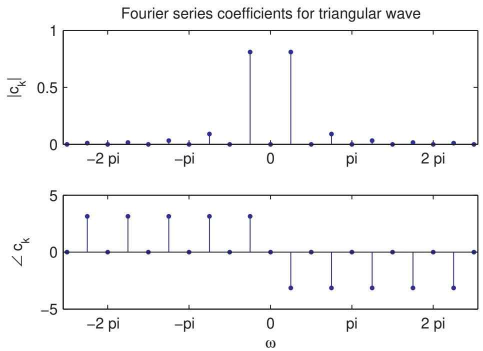
$$c_0 = \frac{1}{8}\int_0^8 x(t)dt = \frac{1}{8}\int_0^4 (t-2)dt + \frac{1}{8}\int_4^8 (4-t)dt = 0.$$

The FS coefficients for $x(t)$ are therefore

$$c_k = \begin{cases} 0 & (k=0) \\ \frac{1}{jk\pi/4}\frac{1}{jk\pi}(1-e^{-jk\pi}) & (k \neq 0). \end{cases}$$

No integration by parts needed!

Triangular wave: FS coefficient plot



Check with Matlab

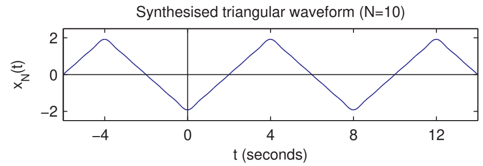
```
j = sqrt(-1);  % to be sure
tv = -6:0.001:14;  % time values (a row vector)
xv = zeros(size(tv));  % signal values initially zero
N = 10;  % highest term in synthesis equation
for k=-N:N
  % Current complex exponential values (a row vector)
  xcv = exp(j*k*pi/4*tv);

  % Coefficient for current complex exponential
  if k==0
    ck = 0;  % DC
  else
    ck = 4/((j*k*pi)^2)*(1 - exp(-j*k*pi));  % formula
    %ck = 1/(j*k*pi)*(1 - exp(-j*k*pi));  % y(t)
    %ck = 1/4*(1 - exp(-j*k*pi));  % z(t)
  end

  % Add scaled complex exponential to signal values
  xv = xv + ck*xcv;
end
plot(tv,real(xv));  % the values *should* be real
```

Exercise 2: Run the above code in Matlab.

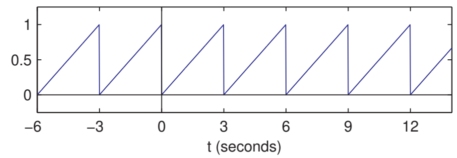
Open Matlab, type `edit fs_reconstruct.m`, press enter. Copy the code above into the editor, save it, and run the file. You will see the reconstruction of the triangular wave using `N` terms. Try

different values N. Look at the code and find the line that calculates the formula, and try the formulas for the two derivative signals. What's going on for the impulse train signal?

Triangular wave: synthesis



Exercise 3: Find the FS of the signal below:



Notes

Some interesting observations:

1. Triangular waveform is continuous (quite smooth), but has discontinuous derivatives — FS coefficients decreases as $\frac{1}{\omega^2}$ — they decrease very quickly as frequency increases.
2. Rectangular waveform is discontinuous (less smooth than triangular waveform) — FS coefficients decreases as $\frac{1}{\omega}$ — more slowly.
3. Smooth waveforms generally contain less high frequency components — their coefficients go to zero closer to the origin in the frequency domain.
4. Alternatively, to reconstruct a signal with fast variations (i.e. discontinuities) requires large components at high frequencies.

Linking coefficients for other transformations

Consider a signal $x(t)$ with period T: it has a FS expansion

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with $\omega_0 = 2\pi/T$.

Let $y(t)$ be a signal related to $x(t)$. If $y(t)$ is also periodic with period T then it has a FS expansion

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}.$$

and the coefficients c_k and d_k are related.

Relationship for shift transformation

Consider the case where $y(t) = x(t-a)$ for some a. Clearly $y(t)$ is periodic with the same period as $x(t)$, and

$$y(t) = x(t-a) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-a)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jk\omega_0 a}$$
$$= \sum_{k=-\infty}^{\infty} (c_k e^{-jk\omega_0 a}) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}.$$

The coefficients are therefore related by

$$d_k = c_k e^{-jk\omega_0 a}$$

Thus a shift corresponds to a change in the phase of each coefficient, where the amount of the change is proportional to the size of the shift.

Exercise 4: Find the FS of the signal below, using the series coefficients c_k found earlier for $x(t)$: