University of Cape Town
EEE515Z exam (Statistical Signal Processing)

Examiner: Fred Nicolls 2 April 1998
Time: 24 hours
Marks: 80

Instructions:

- There are 6 questions in this examination. All of them must be answered.
- This is a take-home exam. The questions may be collected around midday of the day the examination is scheduled for, and should be returned at the same time the following day.
- There is no restriction on material which may be used. Nonetheless, there should be no need for any peripheral information.
- If you get stuck on any aspect of a problem, describe what you are trying to do and continue from there. *A clear demonstration that you understand the concepts is far more important than technical accuracy.*
- All answers should be complete. Any assumptions made must be explicitly stated.
- If you choose to collaborate with others, do so *before* formulating your solution. Solutions which (in my opinion) are unreasonably similar to one another will be penalised. Conversely, answers which are clearly original will be marked with leniency. *Share ideas, not answers.*

Answer all of the following questions

Problem 1 (UMP test of distribution variance) Find the uniformly most powerful test of size 0.1 for testing $H_0 : \sigma^2 \leq 1$ versus $H_1 : \sigma^2 > 1$ in the model

$$\mathbf{x} : N[\mathbf{0}, \sigma^2\mathbf{I}],$$

where $\mathbf{x}$ is 2-dimensional.

Note that if $X_1, \ldots, X_N$ are independent zero-mean normal random variables with variance σ^2, then the probability density function of $T = \sum_{i=1}^{N} X_i^2$ is

$$p_{\sigma^2}(t) = \frac{(t/\sigma^2)^{\frac{N}{2}-1} e^{-\frac{t}{2\sigma^2}}}{2^{N/2}\Gamma(N/2)\sigma^2},$$

where Γ is the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad (\mathrm{Re}(z) > 0).$$

(This probability density is a generalisation of the χ^2 distribution.)

Hint: First reduce the problem by sufficiency.

[15]

Problem 2 (UMP invariant test of distribution location) Let the probability density of the vector (x, y) conditional on the parameter (x_0, y_0) be

$$p_{x_0,y_0}(x, y) = \frac{1}{4} I_{[x_0-1,x_0+1]}(x) I_{[y_0-1,y_0+1]}(y),$$

where $I_{[a,b]}(z)$ is an indicator function such that

$$I_{[a,b]}(z) = \begin{cases} 1 & a \leq z \leq b \\ 0 & \text{otherwise.} \end{cases}$$

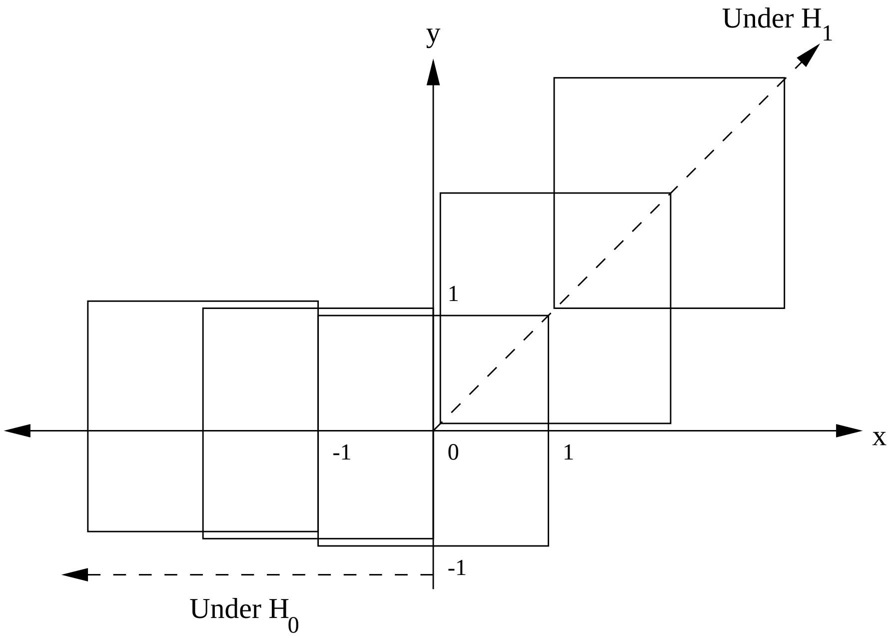
It is assumed that under hypothesis H_0,

$$(x_0, y_0) \in \{(a, b) |\ a \leq 0, b = 0\},$$

and under H_1,

$$(x_0, y_0) \in \{(a, b) |\ a = b, a > 0\}.$$

Informally, we can think of the situation as follows:



Suppose now that every time we actually attempt to observe a vector from this process, the y-component is contaminated by an arbitrary additive contribution for which we have no probability model. The x-component is observed without error. Thus the observation is $(u, v) = (x, y + a)$ for some unknown $a \in \mathbb{R}$, where (x, y) is the actual data realisation.

Formulate a rigorous hypothesis testing problem based on the observation (u, v), taking the uncertainty into account. Clearly specify the density function of (u, v) conditional on the parameters, as well as the parameter spaces under each hypothesis. Suggest a suitable invariance group for the problem, and motivate invariance of the family of distributions and the hypothesis testing problem. Propose a maximal invariant statistic, and formally deduce that it has the required properties (i.e. that it is invariant and maximal). Finally, find the distribution of the maximal invariant, and reformulate the simpler invariant hypothesis testing problem in terms of this statistic.

[15]

Problem 3 (Extension of theoretical concepts) A size α test ϕ of $H_0 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_0$ against $H_1 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_1$ is said to be unbiased if

$$E_{\boldsymbol{\theta}}\phi(\mathbf{x}) \geq \alpha \text{ for all } \boldsymbol{\theta} \in \boldsymbol{\Theta}_1.$$

Explain what this means in terms of probabilities of acceptance and rejection of relevant hypotheses. What do you think is meant by a uniformly most powerful unbiased test, and what is the value of this notion?

[5]

Problem 4 (Application of Bayes' decision rule) Let $x : N[\theta, 1]$, and suppose the hypotheses $H_0 : \theta = 0$, $H_1 : \theta = \theta_1$, and $H_2 : \theta = 2\theta_1$ are in force with equal probabilities. Find the smallest value of θ_1 for which a probability of error of 0.1 can be achieved. Give your answer in terms of the cumulative standard normal distribution

$$G(z) = P\{Z \leq z\} = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

if you have no easy way to find the numerical solution.

[15]

Problem 5 (Estimation of parameters of Gamma distribution) Suppose X is a random variable distributed according to a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$:

$$p_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} U(x),$$

with $\Gamma(z)$ defined as in Problem 1, and

$$U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

We observe n independent realisations $x_1, \ldots, x_n$ of this process, on which we wish to base estimates of the parameters α and β.

a. Find a sufficient statistic for the estimation of the parameters α and β, and justify that the statistic is complete.

b. Find a lower bound on the variance of any unbiased estimates of α and β. It is useful to express the solution in terms of the standard psi function

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

for which tables are available.
c. Find necessary conditions on the maximum likelihood estimates of the parameters α and β. Suggest how you would calculate $\hat{\alpha}$ and $\hat{\beta}$ for any given observation. Any ideas on how to determine whether the estimates are unbiased and/or minimum variance?

(This problem outlines one of the primary objections to maximum likelihood estimation — it is easy to find the estimates numerically, but aside from their asymptotic properties we have no idea how good they are.)

[15]

Problem 6 (Linear prediction) Suppose x_i is the ith element of a scalar random sequence, the samples of which are related by

$$\begin{gathered} x_{i+1} = x_i + u_i \\ Eu_i = 0 \quad Eu_i u_j = \delta_{ij}, \end{gathered}$$

where u_i is independent of x_i. The samples are observed with an independent additive noise component:

$$\begin{gathered} y_k = x_k + v_k \\ Ev_i = 0 \quad Ev_i v_j = \delta_{ij}. \end{gathered}$$

Find the linear minimum mean square error estimate of x_{k+1} given the observations $y_1, \ldots, y_k$. Specifically, if $Ex_1x_1 = 1$, find the estimate of x_4 given $y_1 = 2$, $y_2 = 3$, and $y_3 = 1$.
(Note that the LMMSE estimator is the one, out of the class of *all* estimators of the form $\hat{x}_{k+1} = \sum_{i=1}^{k} a_i y_i$, that has minimum MSE — we do *not* require here that the estimate be unbiased.)

[15]