UNIVERSITY OF CAPE TOWN

EEE515Z EXAM (STATISTICAL SIGNAL PROCESSING)

Examiner: Fred Nicolls 7 April 1997
Time: 3 hours
Marks: 90

Instructions:

- There are 9 questions in this examination. Answer only 6 of them. If you answer more than six, the first six that have been attempted will be assessed.
- This examination is open-book. The only restriction on material is that text books may not be used.
- If you get stuck on a mathematical aspect of a problem, describe what you are trying to do and continue from there.
- Each question is worth 15 marks.

Answer any six of the following nine questions

Problem 1 Find the best test of size 0.2 for testing the hypothesis $H_0 : \mathbf{x} : N[\mathbf{0}, \sigma_0^2\mathbf{I}]$ versus $H_1 : \mathbf{x} : N[\mathbf{0}, \sigma_1^2\mathbf{I}]$ for some 2-D observation $\mathbf{x}$, where $\sigma_0^2 = 1$ and $\sigma_1^2 = 4$.

Note that if $V_t(x_1, x_2)$ is a circular windowing function such that

$$V_t(x_1, x_2) = \begin{cases} 1 & x_1^2 + x_2^2 \leq t \\ 0 & \text{otherwise,} \end{cases}$$

then the following relation holds:

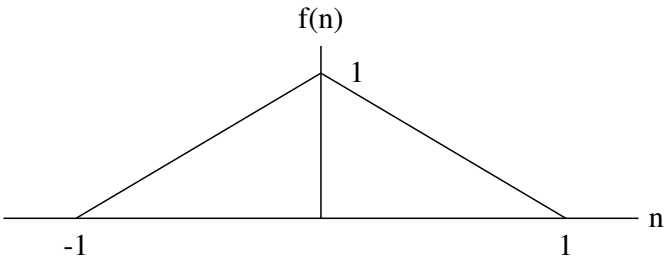
$$\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)} V_t(x_1, x_2) dx_1 dx_2 = 1 - e^{-\frac{t^2}{2\sigma^2}}$$

(this can be found by integrating in radial coordinates).

Problem 2 Find the uniformly most powerful test of size 0.1 for testing $H_0 : \mu = 0$ versus $H_1 : \mu \geq 1$ in the model

$$x = \mu + n,$$

where the distribution of n is



What is the power of the test?

Suppose this detector is implemented and the input data is such that H_0 is in force with probability 0.3. What is the overall false alarm rate at the detector output?

Problem 3 Suppose the observation x has a normal distribution

$$x : N[\mu, 1]$$

where $\mu = -1$ under H_0, $\mu = 0$ under H_1, and $\mu = 1$ under H_2. The hypotheses H_0, H_1, and H_2 are in force with probability 0.25, 0.5, and 0.25 respectively. Find the Bayes rule for testing H_0 versus H_1 versus H_2 if the criterion is to minimise the probability of error.

Problem 4 If N independent and identically distributed observations are made according to the Rayleigh pdf

$$p_{\sigma^2}(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & x \geq 0 \\ 0 & x < 0, \end{cases}$$

find the minimum variance unbiased estimator of σ^2. You will need the result that the sufficient statistic $\mathbf{t}$ is complete in the k-parameter exponential family, and that there is only one function of a complete sufficient statistic which is unbiased. Also note following indefinite integral relation:

$$\int x e^{ax} dx = \frac{1}{a^2} e^{ax}(ax - 1).$$

Problem 5 The observed samples $x_1, x_2, \ldots, x_N$ are independent and identically distributed according to the Laplacian pdf

$$p_\mu(x) = \frac{1}{2} e^{-|x-\mu|}.$$

Find the best linear unbiased estimator of the mean μ. You may need the following relation:

$$\int x^2 e^{ax} dx = \frac{1}{a^3} e^{ax}(a^2x^2 - 2ax + 2).$$

Hint: Propose a general form for the estimator, and minimise the variance under the constraint that the estimator be unbiased.

Problem 6 Given N independent and identically distributed Gaussian observations $x_1, x_2, \ldots, x_N$ with unknown mean μ and unknown variance σ^2.
a. Find the maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}^2$ of these quantities.
b. By calculating the relevant element of the Fisher information matrix, show that $\hat{\mu}$ meets the Cramer-Rao bound, even for the case when σ^2 is unknown.

Problem 7 The conditional distribution of a random variable x is

$$f(x|s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-s)^2}.$$

Assuming s has a Rayleigh distribution

$$f(s) = \begin{cases} se^{-\frac{1}{2}s^2} & s \geq 0 \\ 0 & s < 0, \end{cases}$$

a. Show that the posterior pdf of the parameter s can be written as

$$f(s|x) = \begin{cases} C(x)se^{-(s-\frac{x}{2})^2} & s \geq 0 \\ 0 & s < 0, \end{cases}$$

where all constants not involving s have been lumped into $C(x)$. If we define

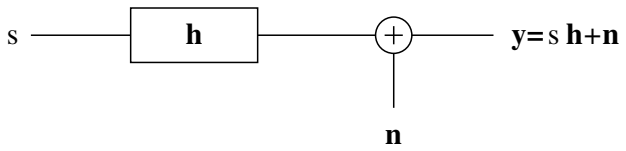
$$F(a) = \int_a^\infty e^{-x^2} dx$$

and

$$G(a) = \int_a^\infty x^2 e^{-x^2} dx,$$

then find $C(x)$ in terms of F and G such that the conditional distribution is normalised.
b. Find the MAP estimate of s.
c. Find the MMSE estimate of s, again in terms of F and G if required.

Problem 8 This question contains two parts. Both must be answered.
a. Consider the measurement system



The random variable s is $N[1,\sigma_s^2]$ and the noise $\mathbf{n}$ is $N[\mathbf{0},\sigma_n^2\mathbf{I}]$, with s and $\mathbf{n}$ independent. Assuming $\mathbf{h}^T\mathbf{h}=1$ and a quadratic loss cost function, find the Bayes estimator of s and its corresponding variance. Use the matrix inversion lemma

$$(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}=\mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\mathbf{H}^{-1}\mathbf{C}\mathbf{A}^{-1},\ \text{where}\ \mathbf{H}=\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B},$$

to simplify the result.
b. Suppose $x_j=f_j s+n_j; j=1,\ldots,m$ with $f_1,\ldots,f_m$ known numbers and s independent of n_j. Given the relations below:

$$E[s]=E[n_j]=0 \qquad E[s^2]=S$$
$$E[n_j^2]=\sigma_n^2 \qquad E[n_i n_j]=0 \quad i\neq j,$$

find the linear minimum mean-square error estimate of s.

Problem 9 The signal model for some system is a first-order Gauss-Markov process

$$\mathbf{s}_n=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\mathbf{s}_{n-1}+\begin{pmatrix}u_n\\ u_n\end{pmatrix} \qquad n\geq 0$$

where $\mathbf{s}_{-1}: N[\mathbf{0},\mathbf{I}]$ and $\sigma_u^2=2$. The first component $(s_1)_n$ of the vector signal $\mathbf{s}_n$ is observed in noise w_n, so the data are $x_n=(s_1)_n+w_n$. w_n are independent samples of Gaussian noise with zero mean and variance $\sigma_n^2=(1/2)^n$. If w_n is independent of $\mathbf{s}_{-1}$ and u_n for $n\geq 0$, use the Kalman filtering formulation to find the best predicted value of the signal $\mathbf{s}_1$ given the observation x_0. What is the variance associated with each component of this vector estimate?