S/N performance of PCM

We know how to find the error probability associated with the transmission of a single bit. However, often it is not the *bit* error rate that is of importance, rather the error in the message that we receive.

Suppose a signal is sampled at a point, and has been quantised to $n = 2^m$ levels. The amplitude increment is a. We assume that m-bit codewords are formed by simply representing the level as a binary string in base-2.

If a linear quantisation rule is used, then an error in the zeroth bit corresponds to an amplitude error of a. An error in the next bit corresponds to $2a$, and an error in the pth bit corresponds to an amplitude error of $(2^p a)$.

If a single bit is received in error in a codeword, the mean square error is

$$\overline{(\Delta m)^2} = \frac{1}{m}[(a^2) + (2a)^2 + (4a)^2 + \cdots + (2^{m-1}a)^2] = \frac{2^{2m} - 1}{3m} a^2.$$

Since the probability that a bit is in error is P_ϵ, the probability that a message is in error is approximately $m P_\epsilon$ (assuming only one bit error per word), so the mean square error per codeword is

$$\overline{n_{\text{th}}^2(t)} = m P_\epsilon \overline{(\Delta m)^2} = P_\epsilon (2^{2m} - 1) a^2 / 3.$$

Including the quantisation error, the SNR is

$$\frac{S_0}{N_0} = \frac{\overline{s^2(t)}}{\overline{n_{\text{qnt}}^2(t)} + \overline{n_{\text{th}}^2(t)}} = \frac{n^2 - 1}{1 + 4P_\epsilon (n^2 - 1)}.$$

See Stremler for a plot of the SNR performance in the case of a polar matched filter receiver.

In the discussion just presented, it is assumed that the PCM is unencoded. By adding additional bits the error rate can be reduced considerably — codewords can easily be designed which can be corrected in the event of only a single bit error.