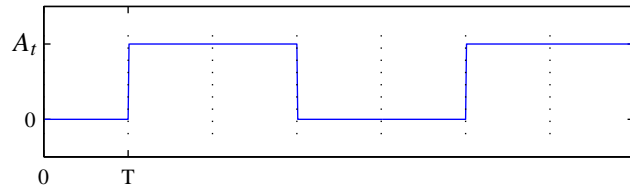


Probability of error in transmission

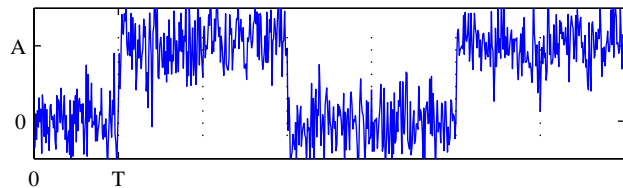
In a binary PCM system, binary digits may be represented by two pulse levels. If these levels are chosen to be 0 and A , the signal is termed an **on-off** binary signal. If the level switches between $-A/2$ and $A/2$ it is called a **polar** binary signal.

Suppose we are transmitting digital information, and decide to do this using two-level pulses each with period T :



The binary digit 0 is represented by a signal of level 0 for the duration T of the transmission, and the digit 1 is represented by the signal level A_T .

In what follows we do not consider modulating the signal — it is transmitted at baseband. In the event of a noisy Gaussian channel (with high bandwidth) the signal at the receiver may look as follows:



Here the binary levels at the receiver are nominally 0 (signal absent) and A (signal present) upon receipt of a 0 or 1 digit respectively.

The function of a receiver is to distinguish the digit 0 from the digit 1. The most important performance characteristic of the receiver is the probability

that an error will be made in such a determination.

Consider the received signal waveform for the bit transmitted between time 0 and time T . Due to the presence of noise the actual waveform $y(t)$ at the receiver is

$$y(t) = f(t) + n(t),$$

where $f(t)$ is the ideal noise-free signal. In the case described the signal $f(t)$ is

$$f(t) = \begin{cases} 0 & \text{symbol 0 transmitted (signal absent)} \\ 1 & \text{symbol 1 transmitted (signal present).} \end{cases}$$

In what follows, it is assumed that the transmitter and the receiver are synchronised, so the receiver has perfect knowledge of the arrival times of sequences of pulses. The means of achieving this synchronisation is not considered here. This means that without loss of generality we can always assume that the bit to be received lies in the interval $(0, T)$.

1 Simple detection

A very simple detector could be obtained by sampling the received signal at some time instant T_s in the range $(0, T)$, and using the value to make a decision. The value obtained would be one of the following:

$$\begin{aligned} y(T_s) &= n(T_s) && \text{signal absent} \\ y(T_s) &= A + n(T_s) && \text{signal present.} \end{aligned}$$

Since the value $n(T)$ is random, we cannot decide with certainty whether a signal was present or not at the time of the sample. However, a reasonable rule for the decision of whether a 0 or a 1 was received is the following:

$$\begin{aligned} y(T) &\leq \mu && \text{signal absent — 0 received} \\ y(T) &> \mu && \text{signal present — 1 received.} \end{aligned}$$

The quantity μ is a threshold which we would usually choose somewhere between 0 and A . For convenience we denote $y(T_s)$ by y .

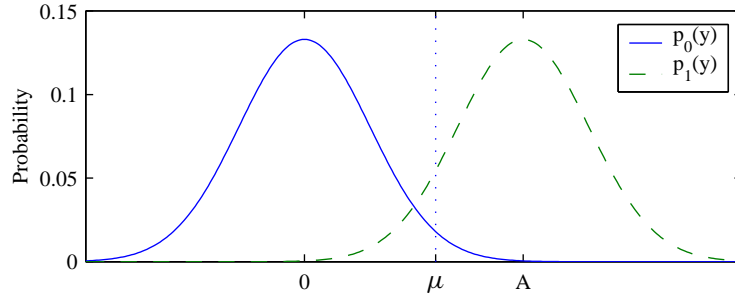
Suppose now that $n(T_s)$ has a Gaussian distribution with a mean of zero and a variance of σ^2 . Under the assumption that a zero was received the probability density of y is

$$p_0(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/(2\sigma^2)}.$$

Similarly, when a signal is present, the density of y is

$$p_1(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-A)^2/(2\sigma^2)}.$$

These are shown below:



Using the decision rule described, it is evident that we sometimes decide that a signal is present even when it is in fact absent. The probability of such a **false alarm** occurring (mistaking a zero for a one) is

$$P_{\epsilon_0} = \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/(2\sigma^2)} dy.$$

Similarly, the probability of a **missed detection** (mistaking a one for a zero) is

$$P_{\epsilon_1} = \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-A)^2/(2\sigma^2)} dy.$$

Letting P_0 and P_1 be the source digit probabilities of zeros and ones

respectively, we can define the overall **probability of error** to be

$$P_{\epsilon} = P_0 P_{\epsilon_0} + P_1 P_{\epsilon_1}.$$

In the equiprobable case this becomes

$$P_{\epsilon} = \frac{1}{2}(P_{\epsilon_0} + P_{\epsilon_1}).$$

The sum of these two errors will be minimised for $\mu = A/2$. This sets the decision threshold for a minimum probability of error for $P_0 = P_1 = \frac{1}{2}$. In that case the probabilities of each type of error are equal, so the overall probability of error is

$$P_{\epsilon} = \int_{A/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/(2\sigma^2)} dy.$$

Making the change of variables $z = y/\sigma$ this integral becomes

$$P_{\epsilon} = \int_{A/(2\sigma)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \text{erfc}\left(\frac{A}{2\sigma}\right).$$

A graph of P_{ϵ} as a function of $A/(2\sigma)$ can be found in Stremler.

This may be written in a more useful form by noting that the average signal power is $S = A^2/2$, and the noise power is $N = \sigma^2$. The probability of error for on-off binary is therefore

$$P_{\epsilon} = \text{erfc}\sqrt{\frac{S}{2N}}.$$

The selection of voltages 0 and A may be difficult for baseband transmission, since an overall DC current flow is implied. If instead a zero is represented by the voltage $-A/2$ and a one by $A/2$, then the entire calculation can be repeated — the only difference is that now $S = (A/2)^2$, so for polar binary

$$P_{\epsilon} = \text{erfc}\sqrt{\frac{S}{N}}.$$

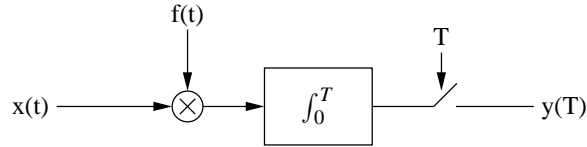
The on-off binary signal therefore requires twice the signal power of the polar binary signal to achieve the same error rate.

2 Matched filter detection

The simple detector just described can be improved upon. Suppose now that the ideal received signal over the interval $(0, T)$ is 0 when the digit 0 is transmitted, and $f(t)$ when 1 is transmitted. Once again there is white Gaussian noise added to the received waveform:

$$\begin{aligned} x(t) &= n(t) && \text{signal absent} \\ x(t) &= f(t) + n(t) && \text{signal present.} \end{aligned}$$

The best detector in this case is the matched filter receiver:

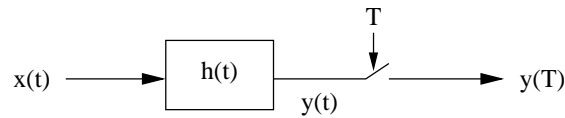


This receiver attempts to recognise the known signal $f(t)$ in the input $x(t)$ by calculating the integral

$$y(T) = \int_0^T x(t) f(t) dt,$$

over the complete signal duration. Recall that the matched filter is the most powerful of all detectors for a known signal in Gaussian noise.

The matched filter can more usefully be viewed as a linear convolution of the input signal, followed by a sampling at the required time instant:



The impulse response of this filter should be chosen as

$$h(t) = \begin{cases} f(T - t) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

to yield the optimal formulation. The result of the convolution is then

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(p) h(t - p) dp = \int_{-\infty}^{\infty} x(p) f(p - (t - T)) dp \\ &= \int_{t-T}^t x(p) f(p - (t - T)) dp, \end{aligned}$$

since $f(t) = 0$ for $t < 0$ and $t > T$. The output at $t = T$ is then

$$y(T) = \int_0^T x(p) f(p) dp,$$

which is the required match filter value. Note also that subsequent samples also give the required integral for later bit intervals: for $t = 2T$,

$$y(2T) = \int_T^{2T} x(p) f(p - T) dp.$$

When the input signal $x(t)$ is identically the signal $f(t)$, the output at $t = T$ is

$$y(T) = \int_0^T |f(t)|^2 dt = E,$$

the energy of the signal. If the input signal is Gaussian white noise with PSD $\eta/2$, then the PSD at the filter output is

$$\frac{\eta}{2} |H(\omega)|^2.$$

The total output power over all frequencies is therefore

$$\sigma_{n_0}^2 = \overline{n_0^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\eta}{2} |H(\omega)|^2 d\omega = \frac{\eta}{2} E$$

The output of the matched filter will be

$$\begin{aligned} y(T) &= n_0(T) && \text{signal absent} \\ y(T) &= E + n_0(T) && \text{signal present.} \end{aligned}$$

Here E is the energy in $f(t)$, or

$$E = \int_0^T f^2(t) dt,$$

and $n_0(T)$ is the component due to noise. This quantity is a zero-mean random variable, and has a mean-square power (variance) equal to

$$\sigma^2 = \overline{n_0^2(t)} = E\eta/2,$$

where η is the noise power spectral density.

Again defining $y = y(T)$, we see that the analysis for the case of simple (sampled) detection continues to apply, except that

- The value A must now be replaced by E , and
- The value σ^2 must be replaced by $\overline{n_0^2(t)} = E\eta/2$.

The probability of error for matched filter detection is therefore

$$P_\epsilon = \text{erfc} \sqrt{\frac{E}{2\eta}}.$$

The polar binary case proceeds in the same way, except that the output of the matched filter is

$$\begin{aligned} y(T) &= -E + n_0(t) && \text{signal absent} \\ y(T) &= E + n_0(T) && \text{signal present.} \end{aligned}$$

Once again we can substitute $2E$ for A and $\sqrt{E\eta/2}$ for σ into the results for the simple case, so

$$P_\epsilon = \text{erfc} \sqrt{\frac{2E}{\eta}}.$$