EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

July 2017
3 hours

Information

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *five* questions totalling 50 marks. You must answer all of them.
- **Part B** has *five* questions totalling 50 marks. You must answer all of them.
- Parts A and B must be answered in different sets of exam books, which will be collected separately.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- You have 3 hours.

PART A

Digital signal processing

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1. Suppose $x[n] = u[n+1]$ and $g[n] = (\frac{1}{2})^n u[n-1]$.
   (a) Sketch $x[n]$ and $g[n]$.
   (b) Find and plot $y[n] = g[n] * x[n]$.
   (c) What is the z-transform $G(z)$? Draw a pole-zero plot and indicate the ROC.
   (d) What is the DC gain of the system with impulse response $g[n]$?

(10 marks)

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2. A causal LTI system is described by the difference equation

$$y[n] = 2.5y[n-1] - 1.5y[n-2] + 3(x[n] - x[n-1]),$$

where $x[n]$ is the input and $y[n]$ is the output.

   (a) Show that the system transfer function is

$$H(z) = \frac{3}{1 - 3/2z^{-1}}.$$

   (b) Draw a pole-zero plot of $H(z)$ and indicate the region of convergence.
   (c) Find the impulse response $h[n]$ of the system.
   (d) Is the system stable?

(10 marks)

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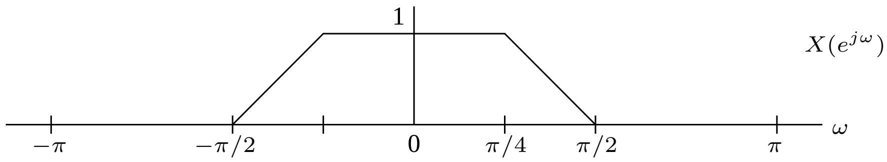
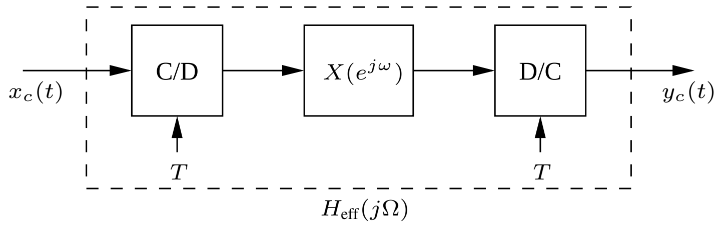
3. Consider the signal below:

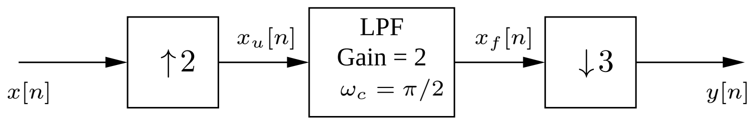


(a) Suppose an analog filter is constructed as follows:



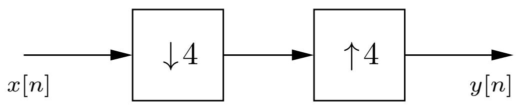
Sketch the frequency response $H_{\text{eff}}(j\Omega)$ for $T = 10^{-3}$ seconds, indicating clearly all frequencies of interest.

(b) Suppose $X(e^{j\omega})$ is the input to the system below:



Sketch $X_u(e^{j\omega})$, $X_f(e^{j\omega})$, and $Y(e^{j\omega})$.

(c) Sketch $Y(e^{j\omega})$ for the case where $X(e^{j\omega})$ is the input to the system below:



(10 marks)

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4. Consider the system function

$$H(z) = 4\frac{z + 0.8}{z^2}.$$

(a) Sketch the frequency response magnitude $H(e^{j\omega})$.

(b) What is the gain of the system for $\omega = 0$ and $\omega = \pi$ rad/sample?

(c) What is the phase of the system for $\omega = \pi/2$ rad/sample?

(d) What type of filter does the system represent?

(e) Find the impulse response of the system.

(10 marks)

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5. Consider two LTI systems with impulse responses

$$h_1[n] = u[-n-1] \qquad \text{and} \qquad h_2[n] = \delta[n-1] - \delta[n].$$

(a) Plot $h_1[n]$ and $h_2[n]$.

(b) Find the system functions $H_1(z)$ and $H_2(z)$ along with their ROCs.

(c) An LTI system with impulse response $h[n]$ is called invertible if there exists $h_i[n]$ such that $h[n] * h_i[n] = \delta[n]$. Are the two systems given above inverses of one another?

(d) Find the impulse response of the inverse system for

$$h[n] = nu[n].$$

(e) What conditions must be poles and zeros of a causal and stable system satisfy for it to have a causal and stable inverse?

(10 marks)

PART B

Wavelets and frames

Problem 1:

Let $\varphi(t)$ denote the unit box function with the support $(0, 1)$. Let $T(t)$ denote the convolution of $\varphi(t)$ with itself. Remember $T(0) = 0$, $T(1) = 0$, and $T(2) = 0$. Let $S(t)$ denote the convolution of $T(t)$ with $\varphi(t)$.

(a) Calculate $S(t)$. (8 marks)

(b) Plot the graph of $S(t)$. (2 marks)

(Total: 10 marks)

Problem 2:

Consider the function $f(t)$ defined as follows:

$$f_n(t) = \begin{cases} \sin(n2\pi \frac{t}{L}) & -\frac{L}{2} \leq t \leq \frac{L}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Calculate $||f_n(t)||_2$, the L_2- norm of $f_n(t)$ for arbitrary n. (3 marks)

(b) Let $\tilde{f}_n(t)$ refer to the normalized $f_n(t)$. Calculate the inner product $< \tilde{f}_m(t)|\tilde{f}_n(t) >$ in detail. (3 marks)

(c) Express the resolution of identity operator $\hat{\mathbb{I}}$ in terms of the orthonormal basis $\{\tilde{f}_n(t)|n \in \hat{\mathbb{N}}\}$. (4 marks)

(Total: 10 marks)

Problem 3:

Let the coefficients h_0, h_1, h_2, h_3 be defined as follows:

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \qquad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \qquad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \qquad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Consider the following dilation equation for the scaling function $\varphi(t)$ with its support being the interval $[0, 3]$:

$$\varphi(t) = h_0\left\{\sqrt{2}\varphi(2t)\right\} + h_1\left\{\sqrt{2}\varphi(2t-1)\right\} + h_2\left\{\sqrt{2}\varphi(2t-2)\right\} + h_3\left\{\sqrt{2}\varphi(2t-3)\right\}.$$

(a) Sample (evaluate) the functions at both sides of this equation successively at integer points $t = 0$, $t = 1$, $t = 2$, and $t = 3$. This process generates a 4×4 eigenvalue equation for the determination of $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, and $\varphi(3)$. Provide the explicit expression for the resulting eigenvalue equation. (2 marks)

(b) From the resulting 4×4 system of equations determine the values $\varphi(0)$ and $\varphi(3)$ first. Then determine the values for $\varphi(1)$ and $\varphi(2)$. (2 marks)

(c) Using the equation above determine the values for $\varphi(\frac{1}{2})$, $\varphi(\frac{3}{2})$, and $\varphi(\frac{5}{2})$. (2 marks)

Consider the following dilation equation for the wavelet $\psi(t)$ with its support being the interval $[-1, 2]$:

$$\psi(t) = -h_0\left\{\sqrt{2}\varphi(2t-1)\right\} + h_1\left\{\sqrt{2}\varphi(2t)\right\} - h_2\left\{\sqrt{2}\varphi(2t+1)\right\} + h_3\left\{\sqrt{2}\varphi(2t+2)\right\}.$$

(d) Calculate the values for $\psi(-1)$, $\psi(-\frac{1}{2})$, $\psi(\frac{1}{2})$, and $\psi(2)$. (4 marks)

(Total: 10 marks)

Problem 4:

(a) Do the discrete filter coefficients $\left\{a_0 = \frac{1}{2\sqrt{2}}, a_1 = \frac{1}{\sqrt{2}}, a_2 = \frac{1}{2\sqrt{2}}\right\}$ constitute a low-pass, bandpass, or a high-pass filter? (2 marks)

(b) Why? (Detailed calculations and a graph are required.) (3 marks)

(c) Do the discrete filter coefficients $\left\{b_0 = -\frac{1}{2\sqrt{2}}, b_1 = \frac{1}{\sqrt{2}}, b_2 = -\frac{1}{2\sqrt{2}}\right\}$ constitute a low-pass, bandpass, or a high-pass filter? (2 marks)

(d) Why? (Detailed calculations and a graph are required.) (3 marks)

(Total: 10 marks)

Problem 5:

Consider the frame vectors $\mathbf{f_1}$, $\mathbf{f_2}$, $\mathbf{f_3}$, and $\mathbf{f_4}$ defined as follows:

$$\mathbf{f_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{f_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{f_3} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{f_4} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) Construct the frame operator $\mathbf{S}$ corresponding to the given frame vectors $\mathbf{f_1}$, $\mathbf{f_2}$, $\mathbf{f_3}$, and $\mathbf{f_4}$. (3 marks)

(b) Construct the dual frame vectors $\tilde{\mathbf{f}}_1$, $\tilde{\mathbf{f}}_2$, $\tilde{\mathbf{f}}_3$, and $\tilde{\mathbf{f}}_4$. (3 marks)

(c) Express the resolution of identity (operator) in terms of the given frame vectors $\mathbf{f_1}$, $\mathbf{f_2}$, $\mathbf{f_3}$, and $\mathbf{f_4}$ and the constructed dual frame vectors $\tilde{\mathbf{f}}_1$, $\tilde{\mathbf{f}}_2$, $\tilde{\mathbf{f}}_3$, and $\tilde{\mathbf{f}}_4$. (2 marks)

(d) Assuming a general vector

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

demonstrate the analysis and synthesis steps by using the constructed expression for the identity operator. (2 marks)

(Total: 10 marks)

Discrete-time Fourier transform properties

Sequences $x[n], y[n]$	Transforms $X(e^{j\omega}), Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common discrete-time Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Z-transform properties

Sequences $x[n], y[n]$	Transforms $X(z), Y(z)$	ROC	Property
$ax[n] + by[n]$	$aX(z) + bY(z)$	ROC contains $R_x \cap R_y$	Linearity
$x[n - n_d]$	$z^{-n_d} X(z)$	ROC $= R_x$	Time shift
$z_0^n x[n]$	$X(z/z_0)$	ROC $= \lvert z_0\rvert R_x$	Frequency scale
$x^*[-n]$	$X^*(1/z^*)$	ROC $= \frac{1}{R_x}$	Time reversal
$nx[n]$	$-z\frac{dX(z)}{dz}$	ROC $= R_x$	Frequency diff.
$x[n] * y[n]$	$X(z)Y(z)$	ROC contains $R_x \cap R_y$	Convolution
$x^*[n]$	$X^*(z^*)$	ROC $= R_x$	Conjugation

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$