

# EEE4001F EXAM

## DIGITAL SIGNAL PROCESSING

University of Cape Town  
Department of Electrical Engineering

June 2013  
3 hours

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### Information

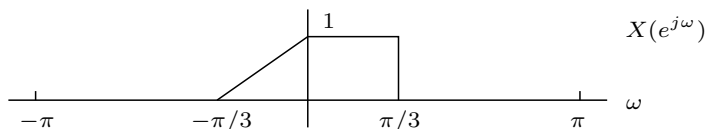
- The exam is closed-book.
  - There are two parts to this exam.
  - **Part A** has *six* questions totalling 50 marks. You must answer all of them.
  - **Part B** has *ten* questions totalling 50 marks. You must answer all of them.
  - Parts A and B must be answered in different sets of exam books, which will be collected separately.
  - A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
  - You have 3 hours.
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## PART A

Basic digital signal processing theory.

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1. A sequence  $x[n]$  has a zero-phase DTFT  $X(e^{j\omega})$  given below:



Sketch the DTFT of the sequence  $2x[n]e^{-j\pi n/3}$ .

(5 marks)

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2. Find the impulse response corresponding to the system function

$$H(z) = \frac{7z^2 - 4z}{z^2 - \frac{3}{2}z - 1}$$

for each possible region of convergence. In each case comment on the causality and stability properties of the system.

(10 marks)

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3. Let  $x[n]$  be a discrete-time signal obtained by sampling the continuous signal  $x(t)$  at a sampling rate  $f_s = 1/T$  Hz:

$$x[n] = x(nT).$$

Assume that no aliasing occurs. Describe by sketching a block diagram and providing a clear explanation, how you would implement a *discrete-time* system with input  $x[n]$  and output  $y[n]$  that delays  $x[n]$  by half a sample, so

$$y[n] = x(nT - T/2).$$

*Hint:* make use of upsamplers, downsamplers, and ideal filters.

(5 marks)

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4. Consider the following discrete-time sequences:

$$x[n] = 2\delta[n] + 3\delta[n - 1] + \delta[n - 2] + 5\delta[n - 3]$$

$$y[n] = \delta[n] - 2\delta[n - 3].$$

- (a) Write an expression for the 4-point DFT  $X[k]$  of  $x[n]$ , and find the value  $X[1]$ .
- (b) Find the 4-point inverse DFT of  $X[k]W_4^k$ , where  $X[k]$  is the 4-point DFT of  $x[n]$  and  $W_4 = e^{-j\frac{2\pi}{4}}$ .
- (c) Find the 4-point circular convolution of  $x[n]$  with  $y[n]$ .
- (d) Explain how you would calculate the result for part (c) using the fast Fourier transform (FFT).
- (e) How could you use the FFT to calculate the linear convolution of  $x[n]$  and  $y[n]$ ?

(10 marks)

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5. A discrete-time, causal, linear time-invariant filter  $H(z)$  has

six zeros located at:  $z = e^{\pm j\pi/8}, z = e^{\pm j7\pi/8}, z = \pm 1$

and six poles located at:  $z = \pm j0.95, z = 0.95e^{\pm j9\pi/20}, z = 0.95e^{\pm j11\pi/20}$ .

- (a) Plot the pole-zero diagram of  $H(z)$  in the  $z$ -plane and provide its region of convergence.
- (b) Sketch the magnitude response  $|H(e^{j\omega})|$  directly from the pole-zero plot, and indicate the approximate gain at  $\omega = \pi/2$ .
- (c) What type of frequency-selective filter is  $H(e^{j\omega})$ ? Explain your answer.
- (d) Answer the following questions, explaining your answers:
  - i. Is  $H(z)$  an IIR or FIR filter?
  - ii. Is  $H(z)$  a stable filter?
  - iii. Is  $h[n]$ , the impulse response of the filter, a real function?

(10 marks)

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6. Consider a causal LTI system with the system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}}$$

where  $a$  is real.

- (a) Write a difference equation that relates the input and the output of this system
- (b) For what range of values of  $a$  is the system stable?
- (c) For  $a = 1/2$  plot the pole-zero diagram and show the ROC.
- (d) Find the impulse response  $h[n]$  for this system for  $a = 1/2$ .
- (e) Determine and plot the magnitude response of this system for  $a = 1/2$ . What type of system is it?

(10 marks)

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## PART B

Wavelets and frames.

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**P1:** Let the function  $f(t)$  be defined by

$$f(t) = \begin{cases} \sin(n\pi t) & -1 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \quad (1)$$

**P1-a:** Calculate  $\|f(t)\|_2$ , the  $L_2$ -norm of  $f(t)$ .

(2 marks)

**P1-b:** Let  $\tilde{f}(t)$  denote  $f(t)$  normalized; i.e.,  $\|\tilde{f}(t)\|_2 = 1$ . Write down the expression for  $\tilde{f}(t)$ .

(2 marks)

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**P2:** Consider the following complete set of orthonormal functions on the interval  $(-1, 1)$ :

$$\frac{1}{\sqrt{2}}, \{\cos(n\pi t) | n \in \mathbb{N}\}, \{\sin(n\pi t) | n \in \mathbb{N}\} \quad (2)$$

Utilizing Dirac's bracket notation, consider the following resolution of identity for the  $L_2$ -space of functions with support  $(-1, 1)$ :

$$\begin{aligned} \mathbb{I} = \left| \frac{1}{\sqrt{2}} \right\rangle \langle \frac{1}{\sqrt{2}} | &+ \sum_{n \in \mathbb{N}} \left| \cos(n\pi t) \right\rangle \langle \cos(n\pi t) | \\ &+ \sum_{n \in \mathbb{N}} \left| \sin(n\pi t) \right\rangle \langle \sin(n\pi t) | \end{aligned} \quad (3)$$

Let the function  $f(t)$  satisfy Dirichlet's conditions on the interval  $(-1, 1)$ , and be zero outside this interval. In bracket notation write  $|f(t)\rangle$  for  $f(t)$ . Operate (3) from the left onto the function  $|f(t)\rangle$  to obtain:

$$\begin{aligned} \mathbb{I}|f(t)\rangle = \left| \frac{1}{\sqrt{2}} \right\rangle \langle \frac{1}{\sqrt{2}} | f(t)\rangle &+ \sum_{n \in \mathbb{N}} \left| \cos(n\pi t) \right\rangle \langle \cos(n\pi t) | f(t)\rangle \\ &+ \sum_{n \in \mathbb{N}} \left| \sin(n\pi t) \right\rangle \langle \sin(n\pi t) | f(t)\rangle \end{aligned} \quad (4)$$

It is self-evident that certain groups of terms in (4) vanish for even- or odd-functions, and thus the Eq. (4) simplifies for such functions.

**P2-a:** Simplify the expression on the right-hand side of the Eq. (4) for functions  $f(t)$  satisfying the condition  $f(-t) = f(t)$  on the interval  $(-1, 1)$ . (2 marks)

**P2-b:** Deduce from your result obtained in **P2-a** the expression for the resolution of identity, which characterizes the space of even functions with support  $(-1, 1)$ . (2 marks)

**P2-c:** Simplify the expression on the right-hand side of the Eq. (4) for functions  $f(t)$  satisfying the condition  $f(-t) = -f(t)$  on the interval  $(-1, 1)$ . (2 marks)

**P2-d:** Deduce from your result obtained in **P2-c** the expression for the resolution of identity, which characterizes the space of odd functions with support  $(-1, 1)$ . (2 marks)

**P3:** Let the functions  $\varphi(t)$  and  $\psi(t)$  denote the scaling function and the wavelet for a Multiresolution Analysis (MRA) in Hilbert space. Let the function  $\varphi(t)$  generate the function space  $\nu_0$ . Let the function  $\psi(t)$  and its compressed versions generate the spaces  $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \dots$ .

Assume the following representation for  $f(t)$  is valid:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi(t - k) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d_{j,k} 2^{\frac{j}{2}} \psi(2^j t - k) \quad (5)$$

Consider the right-hand side of (5).

**P3-a:** Why is the first term a single series?

**P3-b:** Why is the second term (following the summation sign) a double series?

**P3-c:** Write down the expression for  $c_k$ .

**P3-d:** Write down the expression for  $d_{j,k}$ .

**P3-e:** Utilize Dirac's bracket notation, and consider the results obtained in **P3-c** and **P3-d**. In the light of your results deduce the expression for the resolution of identity from (5).

**P3-f:** Let  $j$  run from  $-\infty$  to  $\infty$ . Considering the result obtained in the previous step, deduce the expression for the resolution of identity, when  $j$  varies from  $-\infty$  to  $\infty$ .

(6 marks)

**P4-a:** Given the general ‘‘low pass’’ filter coefficient  $h(n)$  write down the two-scale dilation equation for the scaling function.

(2 marks)

**P4-b:** Given the general ‘‘high pass’’ filter coefficients  $g(n)$  write down the two-scale dilation equation for the wavelet.

(2 marks)

**P5-a:** Determine the filter coefficients  $h(n)$  for the triangle (piece-wise linear) scaling function. (2 marks)

**P5-b:** The coefficients  $h(n)$ , characterizing the triangle (piece-wise linear) scaling function, constitute a “low pass” filter. Why? (2 marks)

**P5-c:** Determine the filter coefficients  $g(n)$  for the triangle (piece-wise linear) wavelet. (2 marks)

**P5-d:** The coefficients  $g(n)$ , characterizing the triangle (piece-wise linear) wavelet, constitute a “high pass” filter. Why? (2 marks)

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**P6:** Given a fairly general function  $f(t)$ . Apply Meyer’s orthogonalization technique to  $f(t)$ . (4 marks)

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**P7:** Using the general dilation equation for the wavelet  $\psi(t)$ , express the three-generations compressed normalized wavelet

$$2^{\frac{3}{2}}\psi(2^3t - m)$$

in terms of  $2^{\frac{4}{2}}\varphi(2^4t - n)$  summed over  $n$ .

(4 marks)

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**P8:** The Mexican-hat wavelet  $\mathcal{M}_h(t)$  can be obtained by taking the second derivative of the negative Gaussian function:

$$\mathcal{G}(t) = -\frac{1}{2}e^{-t^2}$$

Sketch the Mexican-hat wavelet  $\mathcal{M}_h(t)$ .

(2 marks)

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**P9:** Ordinarily signal-analysis and signal -synthesis are carried out by using a system of orthonormal basis (ONB) functions. However, if the orthonormality condition of the analysis basis functions is violated, a system of dual basis functions is required for accomplishing the synthesis of signals. The following problem illustrates the content of this concept in terms of vectors.

Let  $|e_1 \rangle$  and  $|e_2 \rangle$  be unit normal vectors in the  $(x, y)$ -plane.

Let the vectors  $|f_1 \rangle$  and  $|f_2 \rangle$  be defined by the following equations:

$$|f_1 \rangle = 2|e_1 \rangle + |e_2 \rangle \quad (6a)$$

$$|f_2 \rangle = 2|e_1 \rangle + 4|e_2 \rangle \quad (6b)$$

Evidently, the vectors  $|f_1 \rangle$  and  $|f_2 \rangle$  are neither normal nor orthogonal.

Provide a sketch of the vectors  $|f_1 \rangle$  and  $|f_2 \rangle$ .

Construct the dual vectors  $\langle \tilde{f}_1 |$  and  $\langle \tilde{f}_2 |$  corresponding to  $|f_1 \rangle$  and  $|f_2 \rangle$ , respectively, first graphically and then analytically.

Employ Dirac's bracket notation.

Resolve the identity operator  $\mathbb{I}$  in the plane (i.e., the  $2 \times 2$  unity matrix) in terms of the ket-vectors  $|f_1 \rangle$  and  $|f_2 \rangle$  and their dual bra-vectors  $\langle \tilde{f}_1 |$  and  $\langle \tilde{f}_2 |$ .

(3 marks)

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**P10:** In the foregoing problem it was mentioned that customarily signal-analysis and signal-synthesis are carried out by using a system of orthonormal basis (ONB) functions. However, if the analysis functions are over-complete (they constitute a frame), a system of over-complete functions (dual frames) is required for accomplishing the synthesis of signals. The following problem illustrates the content of this concept in terms of vectors. Let  $|e_1 \rangle$  and  $|e_2 \rangle$  denote unit normal vectors in the  $(x, y)$ -plane.

Let the ket vectors  $|f_1 \rangle$ ,  $|f_2 \rangle$  and  $|f_3 \rangle$  be defined by the following equations:

$$|f_1 \rangle = |e_1 \rangle \quad (7a)$$

$$|f_2 \rangle = |e_1 \rangle - |e_2 \rangle \quad (7b)$$

$$|f_3 \rangle = |e_1 \rangle + |e_2 \rangle \quad (7c)$$

The over-complete set of vectors  $|f_1 \rangle$ ,  $|f_2 \rangle$  and  $|f_3 \rangle$  constitutes a frame.

The dual frame (bra vectors)  $\langle \tilde{f}_1 |$ ,  $\langle \tilde{f}_2 |$  and  $\langle \tilde{f}_3 |$  are given as follows:

$$\langle \tilde{f}_1 | = \frac{1}{3} \langle e_1 | \quad (8a)$$

$$\langle \tilde{f}_2 | = \frac{1}{3} \langle e_1 | - \frac{1}{2} \langle e_2 | \quad (8b)$$

$$\langle \tilde{f}_3 | = \frac{1}{3} \langle e_1 | + \frac{1}{2} \langle e_2 | \quad (8c)$$

Resolve the identity operator  $\mathbb{I}$  in the plane (i.e., the  $2 \times 2$  unity matrix) in terms of the frame vectors  $|f_1 \rangle$ ,  $|f_2 \rangle$  and  $|f_3 \rangle$  and their dual frame vectors  $\langle \tilde{f}_1 |$ ,  $\langle \tilde{f}_2 |$  and  $\langle \tilde{f}_3 |$ .

(7 marks)

## Fourier transform properties

Sequences $x[n], y[n]$	Transforms $X(e^{j\omega}), Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$	Modulation

## Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
1 $(-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n]$ $( a  < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n + 1)a^n u[n]$ $( a  < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 &  \omega  < \omega_c \\ 0 & \omega_c <  \omega  \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

## Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All $z$
$u[n]$	$\frac{1}{1 - z^{-1}}$	$ z  > 1$
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z  < 1$
$\delta[n - m]$	$z^{-m}$	All $z$ except 0 or $\infty$
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
$\begin{cases} a^n & 0 \leq n \leq N - 1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z  > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$ z  > r$