EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2009
3 hours

Information

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *eight* questions totalling 70 marks. You must answer all of them.
- **Part B** has *three* questions, each counting 15 marks. You must answer *two* of them.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- A formula sheet for the radar/sonar question appears at the end of this paper.
- You have 3 hours.

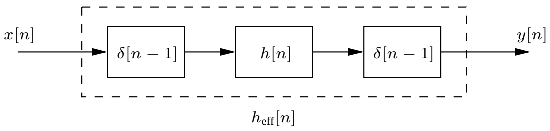
PART A

Answer all of the following questions.

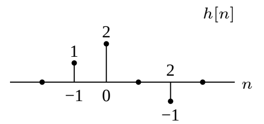
1. Consider the signal $x[n] = \delta[n+1] - \delta[n-1] + 2\delta[n-2]$. Plot the following:
   (a) $y_1[n] = x[-n+2]$
   (b) $y_2[n] = x[n] - x[n-1]$
   (c) $y_3[n] = \sum_{k=-\infty}^{n} x[k]$
   (d) $y_4[n] = x[n] * u[n]$
   (e) $y_5[n] = x[n] * \delta[n-1] * \delta[n+2]$

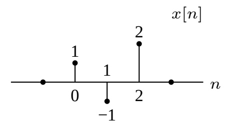
   (10 marks)

2. You are given a system



   with $x[n]$ and $h[n]$ as follows:





   (a) Find and sketch the effective impulse response $h_{\text{eff}}[n]$ linking the input $x[n]$ with the output $y[n]$.
   (b) Is the overall system causal? Why?
   (c) Use time domain convolution to find $y[n]$.

   (10 marks)

3. Use transform properties to show that the DTFT of the sequence $x[n] = (n+1)\alpha^n u[n]$ for $|\alpha| < 1$ is

$$X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}.$$

   (6 marks)

4. The transfer function of an LTI system is

$$H(z) = \frac{z}{(z-0.75)(z+0.5)}.$$

For each possible region of convergence, find the corresponding impulse response $h[n]$ of the system. For which ROC is the system causal, and for which is it stable?

(10 marks)

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5. Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$.

(a) Express the inverse DTFT $y[n]$ of $Y(e^{j\omega}) = X(e^{j4\omega})$ in terms of $x[n]$.

(b) If $y[n] = x[2n]$, specify $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$.

(6 marks)

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6. Given a system with the impulse response

$$h[n] = 2(0.7)^n \cos\left(\frac{\pi}{2}n\right) u[n]$$

(a) Show that the system function is

$$H(z) = \frac{2z^2}{(z-0.7j)(z+0.7j)}.$$

(b) Draw the pole-zero plot of the system.

(c) Sketch the magnitude frequency response of the system.

(d) Plot the first 6 values of the output of the system when the input is $x[n] = \delta[n]$.

(10 marks)

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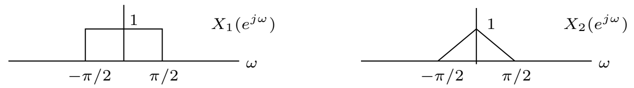
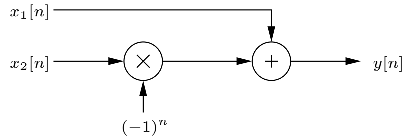
7. Let $x_1[n]$ and $x_2[n]$ be signals with DTFTs given below:



Define the system



with $y[n] = x_1[n] + (-1)^n x_2[n]$.

(a) Sketch $|Y(e^{j\omega})|$ for $-\pi < \omega < \pi$.

(b) Design a system to recover $x_1[n]$ and $x_2[n]$ from $y[n]$. Specify your system in the form of a block diagram, and justify your design. You may use an ideal lowpass filter $H_{\text{LP}}(e^{j\omega})$ with cutoff ω_c (which you specify).

(8 marks)

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8. Consider a filter described by the difference equation

$$y[n] = 2x[n] + 2x[n-1].$$

A periodic input signal $x[n]$ with period 4 is applied to this system, giving the output $y[n]$. The 4-point DFT of $y[n]$ is

$$Y[0] = 0, \quad Y[1] = \sqrt{2}e^{-j\pi/4}, \quad Y[2] = 0, \quad Y[3] = \sqrt{2}e^{j\pi/4}.$$

Find an expression for the input $x[n]$.

(10 marks)

PART B

Answer *two* of the following three questions. Each question counts 15 marks.

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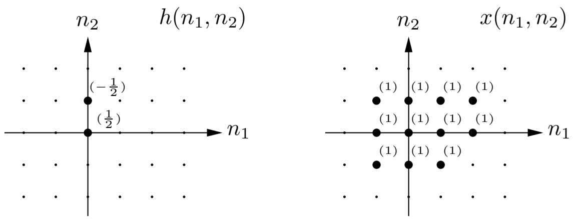
1. **Image processing and computer vision**

Consider the 2D signals below:



(a) Find and sketch the 2-D convolution $h(n_1, n_2) * x(n_1, n_2)$.

(5 marks)

(b) The signal $h(n_1, n_2)$ in the previous question is the impulse response of a 2-D edge detector. Explain why a system with this impulse response is able to detect edges. What edge direction does it detect? Draw the impulse response of a filter that can be used to detect edges in the orthogonal direction. How could you modify these filters to reduce noise in the filter output?

(5 marks)

(c) Find an expression for the 2D discrete-time Fourier transform of the signal $h(n_1, n_2)$. Plot this transform for the two cases of $n_1 = 0$ and $n_2 = 0$. Interpret and explain your results.

(5 marks)

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2. **Radar/sonar signal processing**

(a) Draw a neatly labelled block diagram of a coherent radar system showing (i) transmitter chain with I-Q up converter (ii) receiver with an I-Q down converter (iii) appropriate sampling into a digital signal processor.

(2 marks)

(b) Draw an equivalent "end-to-end" block digram model which relates the impulse response of the scene $\zeta(t)$ (at the input), to the complex baseband signl $v_{bb}(t) = I(t) + jQ(t)$ (at the output).

(2 marks)

(c) Illustrate with the aid of sketches of the <u>frequency</u> spectra, how the signals in the system are related, particularly (i) the impulse response of the scene $\zeta(t) \leftrightarrow \zeta(f)$ (ii); the transmitted rf pulse $v_{tx}(t) \leftrightarrow V_{tx}(f)$ and its baseband form $p(t) \leftrightarrow P(f)$; (iii) complex baseband signal $v_{bb}(t) \leftrightarrow V_{bb}(f)$.

(3 marks)

(d) What properties of the transmitted pulse determine (i) the resolution of the radar (ii) the SNR?

(2 marks)

(e) What are the primary advantages of transmitting a CHIRP pulse as opposed to a monochrome pulse?

(2 marks)

(f) A digital signal processing algorithm must be developed for pulse compression and display of the received echoes. The transmitted pulse is a chirp pulse with bandwidth of 100 MHz. The processor operates on the samplex complex baseband received signal.
(i) What digital signal processing steps would you apply to obtain a profile of the scene, considering that you would like to optimize signal to noise ratio and keep the sidelobes low?
(ii) Sketch the point target response at the output of your processor display, indicating the range resolution achieved.

(4 marks)

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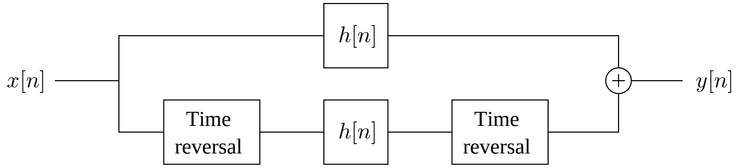
3. **Additional DSP theory**

Let a causal LTI discrete-time system be characterised by a real impulse response $h[n]$ with a DTFT $H(e^{j\omega})$. Consider the system shown below, where $x[n]$ is a finite-length sequence.



Determine the frequency response of the overall system $G(e^{j\omega})$ in terms of $H(e^{j\omega})$, and show that it has a zero phase response. What is the significance of such a zero phase response? What is the significance of the way on which zero phase has been achieved in the above system?

(15 marks)

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

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PLEASE REPORT ANY ERRORS TO A.J.W.

Fourier Relationships

$x(t) \;\leftrightarrow\; X(f)$

$x(t - t_o) \;\leftrightarrow\; X(f)e^{-j2\pi f t_o}$

$x(t)e^{-j2\pi f_o t} \;\leftrightarrow\; X(f + f_o)$

$x^*(t) \;\leftrightarrow\; X^*(-f)$

$BSa(\pi\beta t) \;\leftrightarrow\; rect(\frac{f}{B})$

$rect\left(\frac{t}{\tau}\right) \;\leftrightarrow\; \tau Sa(\pi f\tau)$

$\delta(t) \;\leftrightarrow\; 1$

For any 'real' signal $x(t)$, $X(-f) = X^*(f)$

Convolution $x(t) \otimes h(t) \;\leftrightarrow\; X(f)H(f)$

IQ Down-converter

$I(t) = [2x(t)\cos(\omega_o t)]_{LPF}$

$Q(t) = [-2x(t)\sin(\omega_o t)]_{LPF}$

$V(t) = I(t) + jQ(t) \;\leftrightarrow\; V(f) = 2X^+(f + f_o)$

Matched Filter General

$H(\omega) = \frac{X^*(\omega)}{S_{n_i}(\omega)} \;\rightarrow\; X^*(\omega)$ (white noise)

$\frac{\left|\upsilon_o(t_{peak})\right|^2}{\left|\overline{n_o(t)}\right|^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\left|X(\omega)\right|^2}{S_{n_i}(w)} d\omega \;\rightarrow\; \frac{E}{\eta/2}$ (white noise)

ANALYTIC RADAR MODEL

Baseband Pulse $p(t)$

Transmitted $\upsilon_{TX}(t) = p(t)e^{j2\pi f_o t}$

EXTENDED TARGET RESPONSE

$\upsilon_{RX}(t) = \int_{\tau=-\infty}^{\infty} \zeta(\tau)\upsilon_{TX}(t-\tau)d\tau = \zeta(t) \otimes \upsilon_{TX}(t)$

$\left|\zeta(\tau)\right|^2 \propto \frac{1}{R^4(\tau)}\left|\beta(\tau)\right|^2$

$V_{RX}(f) = \zeta(f)V_{TX}(f)$

Baseband Signal

$\upsilon_{bb}(t) = \left[\upsilon_{RX}(t)e^{-j2\pi f_o t}\right] \otimes h_{bb}(t) + n_{bb}(t)$

$\upsilon_{bb}(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \otimes p(t) \otimes h_{bb}(t) + n_{bb}(t)$

$V_{bb}(f) = \zeta(f + f_o)\, P(f)\, H_{bb}(f) + N_{bb}(f)$

After Deconvolution/Inverse Filter

$V(f) = \zeta(f + f_o)\text{rect}(\frac{f}{B})$

$\upsilon(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \otimes B\frac{\sin(\pi Bt)}{(\pi Bt)}$

where $\frac{\sin(\pi Bt)}{(\pi Bt)} \equiv Sa(\pi Bt)$

POINT TARGET RESPONSE

$\upsilon_{RX}(t) = a_1 \upsilon_{TX}(t-\tau)$ where $\tau = \frac{2R}{c}$

$a_1 \propto \sqrt{\frac{G_t G_r \sigma \lambda^2}{(4\pi)^3 R^4}}$ (narrowband)

$\upsilon_{RX}(t) = \sum_{i=1}^{N} a_i \upsilon_{TX}(t-\tau_i)$ where $\tau_i = \frac{2R_i}{c}$

Baseband

$\upsilon_{bb}(t) = \upsilon_{RX}(t)\, e^{-j\omega_o t} \otimes h_{bb}(t) = \zeta\, p(t-\tau)\, e^{-j\omega_o \tau} \otimes h_{bb}(t)$

After deconvolution filtering

$\upsilon(t) = a_1 B\, Sa(\pi B[t-\tau]) e^{-j2\pi f_o \tau}$

$\psi = arg\left\{e^{-j2\pi f_o \tau}\right\} = arg\left\{e^{-j4\pi R/\lambda}\right\}$

Resolution

$\delta t_{3dB} \approx \frac{0.89}{B} \quad \delta R_{3dB} = \frac{c\delta t_{3dB}}{2} \approx \frac{c}{2B}(0.89)$

Radar Filters

Ideal Spectral Reconstruction (deconvolution/inverse) Filter

$H_{IRF}(f) = \frac{1}{P(f) H_{bb}(f)}$ over $-\frac{B}{2} \leq f \leq \frac{B}{2}$

Matched Filter (MF) $H_{MF}(f) = \frac{P^*(f)}{H_{bb}(f)} \approx P^*(f)$

Doppler Shift $f_D = \frac{-2\, dR/dt}{\lambda}$

MONOCHROME PULSE

RF: $\upsilon_{RF}(t) = rect\left(\frac{t}{T}\right)\cos(2\pi f_o t)$

Analytic: $\upsilon_{TX}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi f_o t}$

Baseband: $\upsilon_{bb}(t) = rect\left(\frac{t}{T}\right)$

Frequency Domain

$V_{TX}(f) = T\frac{\sin(\pi T(f-f_o))}{\pi T(f-f_o)}$

$V_{bb}(f) = T\frac{\sin(\pi T f)}{(\pi T f)}$

LINEAR FM CHIRP

RF signal $\upsilon_{RF}(t) = rect\left(\frac{t}{T}\right)\cos\left(2\pi[f_o t + \frac{1}{2}Kt^2]\right)$

Analytic: $\upsilon_{TX}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi[f_o t + \frac{1}{2}Kt^2]}$

Baseband: $\upsilon_{bb}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi\frac{1}{2}Kt^2}$

Sweep range $\Delta f = KT$ [Hz]

Instantaneous Frequency

RF: $f_{RF}(t) = \frac{1}{2\pi}\frac{d\psi_{RF}(t)}{dt} = f_o + Kt \quad [Hz]$

Baseband: $f_{bb(t)=Kt}$ [Hz]

Dispersion factor $D = \Delta f\, T = KT^2$

Frequency Domain D¿50

$|\upsilon_{bb}(f)| \approx rect\left(\frac{f}{\Delta f}\right)\frac{1}{\sqrt{|K|}}$

$arg\{\upsilon_{bb}(f)\} = W\left\{-j\frac{\pi}{K}f^2\right\}$