

# EEE4001F EXAM

## DIGITAL SIGNAL PROCESSING

University of Cape Town  
Department of Electrical Engineering

June 2009

3 hours

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### Information

- The exam is closed-book.
  - There are two parts to this exam.
  - **Part A** has *eight* questions totalling 70 marks. You must answer all of them.
  - **Part B** has *three* questions, each counting 15 marks. You must answer *two* of them.
  - A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
  - A formula sheet for the radar/sonar question appears at the end of this paper.
  - You have 3 hours.
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## PART A

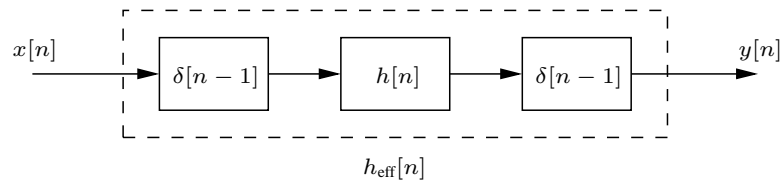
Answer all of the following questions.

1. Consider the signal  $x[n] = \delta[n + 1] - \delta[n - 1] + 2\delta[n - 2]$ . Plot the following:

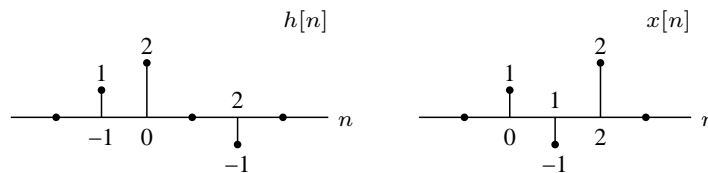
- (a)  $y_1[n] = x[-n + 2]$
- (b)  $y_2[n] = x[n] - x[n - 1]$
- (c)  $y_3[n] = \sum_{k=-\infty}^n x[k]$
- (d)  $y_4[n] = x[n] * u[n]$
- (e)  $y_5[n] = x[n] * \delta[n - 1] * \delta[n + 2]$

(10 marks)

2. You are given a system



with  $x[n]$  and  $h[n]$  as follows:



- (a) Find and sketch the effective impulse response  $h_{\text{eff}}[n]$  linking the input  $x[n]$  with the output  $y[n]$ .
- (b) Is the overall system causal? Why?
- (c) Use time domain convolution to find  $y[n]$ .

(10 marks)

3. Use transform properties to show that the DTFT of the sequence  $x[n] = (n + 1)\alpha^n u[n]$  for  $|\alpha| < 1$  is

$$X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2}.$$

(6 marks)

4. The transfer function of an LTI system is

$$H(z) = \frac{z}{(z - 0.75)(z + 0.5)}.$$

For each possible region of convergence, find the corresponding impulse response  $h[n]$  of the system. For which ROC is the system causal, and for which is it stable?

(10 marks)

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5. Let  $X(e^{j\omega})$  denote the DTFT of a real sequence  $x[n]$ .

(a) Express the inverse DTFT  $y[n]$  of  $Y(e^{j\omega}) = X(e^{j4\omega})$  in terms of  $x[n]$ .

(b) If  $y[n] = x[2n]$ , specify  $Y(e^{j\omega})$  in terms of  $X(e^{j\omega})$ .

(6 marks)

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6. Given a system with the impulse response

$$h[n] = 2(0.7)^n \cos\left(\frac{\pi}{2}n\right) u[n]$$

(a) Show that the system function is

$$H(z) = \frac{2z^2}{(z - 0.7j)(z + 0.7j)}.$$

(b) Draw the pole-zero plot of the system.

(c) Sketch the magnitude frequency response of the system.

(d) Plot the first 6 values of the output of the system when the input is  $x[n] = \delta[n]$ .

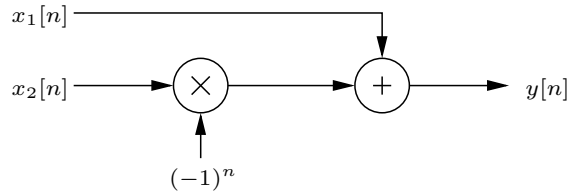
(10 marks)

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7. Let  $x_1[n]$  and  $x_2[n]$  be signals with DTFTs given below:



Define the system



with  $y[n] = x_1[n] + (-1)^n x_2[n]$ .

- Sketch  $|Y(e^{j\omega})|$  for  $-\pi < \omega < \pi$ .
- Design a system to recover  $x_1[n]$  and  $x_2[n]$  from  $y[n]$ . Specify your system in the form of a block diagram, and justify your design. You may use an ideal lowpass filter  $H_{LP}(e^{j\omega})$  with cutoff  $\omega_c$  (which you specify).

(8 marks)

8. Consider a filter described by the difference equation

$$y[n] = 2x[n] + 2x[n - 1].$$

A periodic input signal  $x[n]$  with period 4 is applied to this system, giving the output  $y[n]$ . The 4-point DFT of  $y[n]$  is

$$Y[0] = 0, \quad Y[1] = \sqrt{2}e^{-j\pi/4}, \quad Y[2] = 0, \quad Y[3] = \sqrt{2}e^{j\pi/4}.$$

Find an expression for the input  $x[n]$ .

(10 marks)

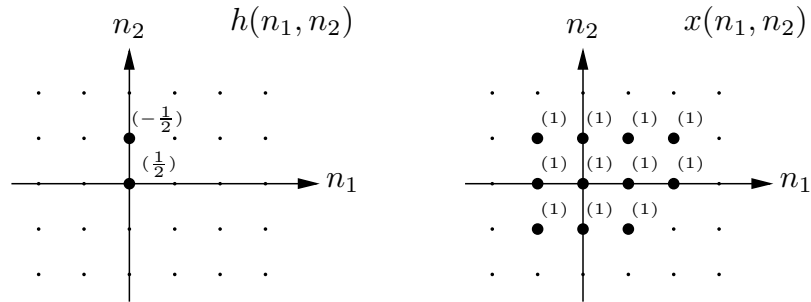
## PART B

Answer *two* of the following three questions. Each question counts 15 marks.

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### 1. Image processing and computer vision

Consider the 2D signals below:



- (a) Find and sketch the 2-D convolution  $h(n_1, n_2) * x(n_1, n_2)$ .

(5 marks)

- (b) The signal  $h(n_1, n_2)$  in the previous question is the impulse response of a 2-D edge detector. Explain why a system with this impulse response is able to detect edges. What edge direction does it detect? Draw the impulse response of a filter that can be used to detect edges in the orthogonal direction. How could you modify these filters to reduce noise in the filter output?

(5 marks)

- (c) Find an expression for the 2D discrete-time Fourier transform of the signal  $h(n_1, n_2)$ . Plot this transform for the two cases of  $n_1 = 0$  and  $n_2 = 0$ . Interpret and explain your results.

(5 marks)

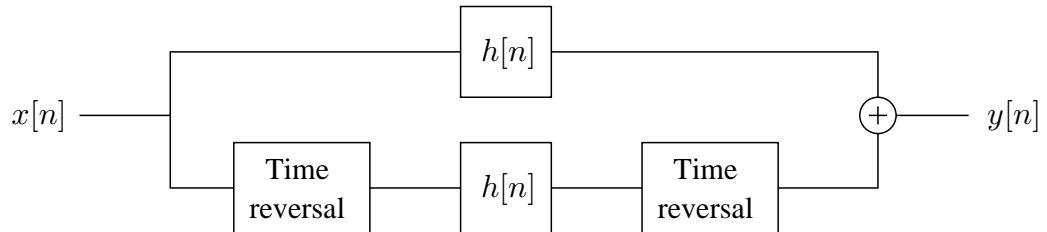
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## 2. Radar/sonar signal processing

- (a) Draw a neatly labelled block diagram of a coherent radar system showing (i) transmitter chain with I-Q up converter (ii) receiver with an I-Q down converter (iii) appropriate sampling into a digital signal processor. (2 marks)
- (b) Draw an equivalent “end-to-end” block diagram model which relates the impulse response of the scene  $\zeta(t)$  (at the input), to the complex baseband signal  $v_{bb}(t) = I(t) + jQ(t)$  (at the output). (2 marks)
- (c) Illustrate with the aid of sketches of the frequency spectra, how the signals in the system are related, particularly (i) the impulse response of the scene  $\zeta(t) \leftrightarrow \zeta(f)$  (ii); the transmitted rf pulse  $v_{tx}(t) \leftrightarrow V_{tx}(f)$  and its baseband form  $p(t) \leftrightarrow P(f)$ ; (iii) complex baseband signal  $v_{bb}(t) \leftrightarrow V_{bb}(f)$ . (3 marks)
- (d) What properties of the transmitted pulse determine (i) the resolution of the radar (ii) the SNR? (2 marks)
- (e) What are the primary advantages of transmitting a CHIRP pulse as opposed to a monochrome pulse? (2 marks)
- (f) A digital signal processing algorithm must be developed for pulse compression and display of the received echoes. The transmitted pulse is a chirp pulse with bandwidth of 100 MHz. The processor operates on the sample complex baseband received signal.
- (i) What digital signal processing steps would you apply to obtain a profile of the scene, considering that you would like to optimize signal to noise ratio and keep the sidelobes low?
- (ii) Sketch the point target response at the output of your processor display, indicating the range resolution achieved. (4 marks)
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### 3. Additional DSP theory

Let a causal LTI discrete-time system be characterised by a real impulse response  $h[n]$  with a DTFT  $H(e^{j\omega})$ . Consider the system shown below, where  $x[n]$  is a finite-length sequence.



Determine the frequency response of the overall system  $G(e^{j\omega})$  in terms of  $H(e^{j\omega})$ , and show that it has a zero phase response. What is the significance of such a zero phase response? What is the significance of the way on which zero phase has been achieved in the above system?

(15 marks)

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## Fourier transform properties

Sequences $x[n], y[n]$	Transforms $X(e^{j\omega}), Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$	Modulation

## Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
1 $(-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n]$ $( a  < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n + 1)a^n u[n]$ $( a  < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 &  \omega  < \omega_c \\ 0 & \omega_c <  \omega  \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

## Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All $z$
$u[n]$	$\frac{1}{1 - z^{-1}}$	$ z  > 1$
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z  < 1$
$\delta[n - m]$	$z^{-m}$	All $z$ except 0 or $\infty$
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
$\begin{cases} a^n & 0 \leq n \leq N - 1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z  > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z  > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$ z  > r$

## FORMULA SHEET V4 EEE4001F 2009

PLEASE REPORT ANY ERRORS TO A.J.W.

### Fourier Relationships

$$x(t) \leftrightarrow X(f)$$

$$x(t - t_o) \leftrightarrow X(f)e^{-j2\pi f t_o}$$

$$x(t)e^{-j2\pi f_o t} \leftrightarrow X(f + f_o)$$

$$x^*(t) \leftrightarrow X^*(-f)$$

$$BSa(\pi\beta t) \leftrightarrow \text{rect}\left(\frac{f}{B}\right)$$

$$\text{rect}\left(\frac{t}{\tau}\right) \leftrightarrow \tau Sa(\pi f \tau)$$

$$\delta(t) \leftrightarrow 1$$

For any 'real' signal  $x(t)$ ,  $X(-f) = X^*(f)$

Convolution  $x(t) \otimes h(t) \leftrightarrow X(f)H(f)$

### IQ Down-converter

$$I(t) = [2x(t) \cos(\omega_o t)]_{LPF}$$

$$Q(t) = [-2x(t) \sin(\omega_o t)]_{LPF}$$

$$V(t) = I(t) + jQ(t) \leftrightarrow V(f) = 2X^+(f + f_o)$$

### Matched Filter General

$$H(\omega) = \frac{X^*(\omega)}{S_{n_i}(\omega)} \rightarrow X^*(\omega) \text{ (white noise)}$$

$$\frac{|v_o(t_{peak})|^2}{|n_o(t)|^2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X(\omega)|^2}{S_{n_i}(\omega)} d\omega \rightarrow \frac{E}{\eta/2} \text{ (white noise)}$$

### ANALYTIC RADAR MODEL

#### Baseband Pulse $p(t)$

$$\text{Transmitted } v_{TX}(t) = p(t)e^{j2\pi f_o t}$$

### EXTENDED TARGET RESPONSE

$$v_{RX}(t) = \int_{\tau=-\infty}^{\infty} \zeta(\tau)v_{TX}(t - \tau)d\tau = \zeta(t) \otimes v_{TX}(t)$$

$$|\zeta(\tau)|^2 \propto \frac{1}{R^4(\tau)} |\beta(\tau)|^2$$

$$V_{RX}(f) = \zeta(f)V_{TX}(f)$$

#### Baseband Signal

$$v_{bb}(t) = [v_{RX}(t)e^{-j2\pi f_o t}] \otimes h_{bb}(t) + n_{bb}(t)$$

$$v_{bb}(t) = [\zeta(t)e^{-j2\pi f_o t}] \otimes p(t) \otimes h_{bb}(t) + n_{bb}(t)$$

$$V_{bb}(f) = \zeta(f + f_o)P(f)H_{bb}(f) + N_{bb}(f)$$

#### After Deconvolution/Inverse Filter

$$V(f) = \zeta(f + f_o)\text{rect}\left(\frac{f}{B}\right)$$

$$v(t) = [\zeta(t)e^{-j2\pi f_o t}] \otimes B \frac{\sin(\pi B t)}{(\pi B t)}$$

$$\text{where } \frac{\sin(\pi B t)}{(\pi B t)} \equiv Sa(\pi B t)$$

## POINT TARGET RESPONSE

$$v_{RX}(t) = a_1 v_{TX}(t - \tau) \text{ where } \tau = \frac{2R}{c}$$

$$a_1 \propto \sqrt{\frac{G_t G_r \sigma \lambda^2}{(4\pi)^3 R^4}} \text{ (narrowband)}$$

$$v_{RX}(t) = \sum_{i=1}^N a_i v_{TX}(t - \tau_i) \text{ where } \tau_i = \frac{2R_i}{c}$$

### Baseband

$$v_{bb}(t) = v_{RX}(t) e^{-j\omega_o t} \otimes h_{bb}(t) = \zeta p(t - \tau) e^{-j\omega_o \tau} \otimes h_{bb}(t)$$

### After deconvolution filtering

$$v(t) = a_1 B Sa(\pi B[t - \tau]) e^{-j2\pi f_o \tau}$$

$$\psi = \arg \{ e^{-j2\pi f_o \tau} \} = \arg \{ e^{-j4\pi R/\lambda} \}$$

### Resolution

$$\delta t_{3dB} \approx \frac{0.89}{B} \quad \delta R_{3dB} = \frac{c \delta t_{3dB}}{2} \approx \frac{c}{2B} (0.89)$$

### Radar Filters

Ideal Spectral Reconstruction (deconvolution/inverse) Filter

$$H_{IRF}(f) = \frac{1}{P(f)H_{bb}(f)} \text{ over } -\frac{B}{2} \leq f \leq \frac{B}{2}$$

$$\text{Matched Filter (MF)} H_{MF}(f) = \frac{P^*(f)}{H_{bb}(f)} \approx P^*(f)$$

$$\text{Doppler Shift } f_D = \frac{-2 dR/dt}{\lambda}$$

## MONOCHROME PULSE

$$\text{RF: } v_{RF}(t) = \text{rect}\left(\frac{t}{T}\right) \cos(2\pi f_o t)$$

$$\text{Analytic: } v_{TX}(t) = \text{rect}\left(\frac{t}{T}\right) e^{j2\pi f_o t}$$

$$\text{Baseband: } v_{bb}(t) = \text{rect}\left(\frac{t}{T}\right)$$

### Frequency Domain

$$V_{TX}(f) = T \frac{\sin(\pi T(f - f_o))}{\pi T(f - f_o)}$$

$$V_{bb}(f) = T \frac{\sin(\pi T f)}{(\pi T f)}$$

## LINEAR FM CHIRP

$$\text{RF signal } v_{RF}(t) = \text{rect}\left(\frac{t}{T}\right) \cos\left(2\pi\left[f_o t + \frac{1}{2} K t^2\right]\right)$$

$$\text{Analytic: } v_{TX}(t) = \text{rect}\left(\frac{t}{T}\right) e^{j2\pi\left[f_o t + \frac{1}{2} K t^2\right]}$$

$$\text{Baseband: } v_{bb}(t) = \text{rect}\left(\frac{t}{T}\right) e^{j2\pi\frac{1}{2} K t^2}$$

$$\text{Sweep range } \Delta f = K T \quad [\text{Hz}]$$

### Instantaneous Frequency

$$\text{RF: } f_{RF}(t) = \frac{1}{2\pi} \frac{d\psi_{RF}(t)}{dt} = f_o + K t \quad [\text{Hz}]$$

$$\text{Baseband: } f_{bb(t)=Kt} \quad [\text{Hz}]$$

$$\text{Dispersion factor } D = \Delta f T = K T^2$$

### Frequency Domain $D \geq 50$

$$|v_{bb}(f)| \approx \text{rect}\left(\frac{f}{\Delta f}\right) \frac{1}{\sqrt{|K|}}$$

$$\arg\{v_{bb}(f)\} = W \left\{ -j \frac{\pi}{K} f^2 \right\}$$