EEE4001F: Digital Signal Processing

Class Test 1

20 March 2015

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.
- An information sheet is attached.

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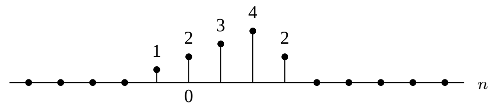
1. (5 marks) Suppose $x[n]$ is as given below:



Plot the following:

(a) $y_1[n] = x[2n]$

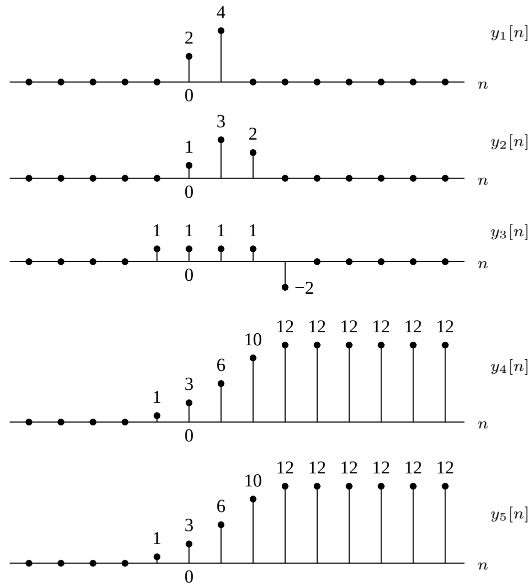
(b) $y_2[n] = x[2n-1]$

(c) $y_3[n] = x[n] - x[n-1]$

(d) $y_4[n] = \sum_{k=-\infty}^{n} x[k]$

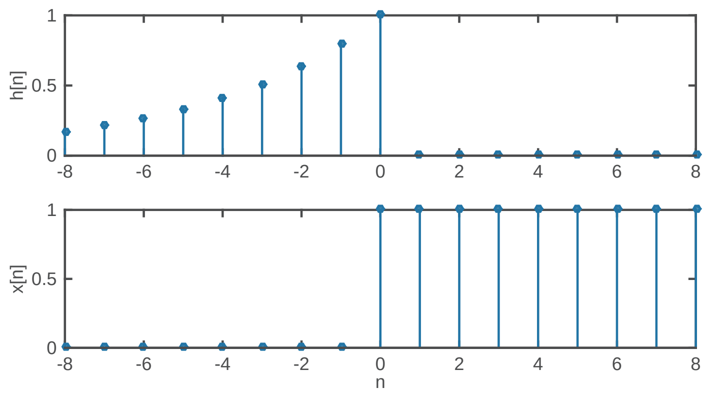
(e) $y_5[n] = x[n] * u[n]$.

Plots as follows:



2. (5 marks) A linear time-invariant system with impulse response $h[n] = a^{-n}u[-n]$ (for $0 < a < 1$) is driven by the unit step input $x[n] = u[n]$. Sketch the signals $h[n]$ and $x[n]$ and find the output $y[n] = h[n] * x[n]$ for values $n = 2$ and $n = -2$.

Plots as follows:



The output is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$
$$= \sum_{k=-\infty}^{0} a^{-k}u[n-k] = \sum_{m=0}^{\infty} a^m u[n+m].$$

Thus

$$y[2] = \sum_{m=0}^{\infty} a^m u[m+2] = \sum_{m=0}^{\infty} a^m = \frac{1}{1-a}$$

and

$$y[-2] = \sum_{m=0}^{\infty} a^m u[m-2] = \sum_{m=2}^{\infty} a^m = \sum_{m=2}^{\infty} a^2 a^{m-2} = a^2 \sum_{m=0}^{\infty} a^m = \frac{a^2}{1-a}.$$

3. (4 marks) Find a closed-form expression for the frequency response $H(e^{j\omega})$ of the FIR filter with impulse response

$$h[n] = a^n(u[n] - u[n-10]).$$

Is the filter causal? Why?

The required transform is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n(u[n] - u[n-10])e^{-j\omega n}$$
$$= \sum_{n=0}^{9} (ae^{-j\omega})^n = \frac{1-(ae^{-j\omega})^N}{1-(ae^{-j\omega})}.$$

The impulse response is right sided ($h[n] = 0$ for $n < 0$) so the system is causal.

4. (6 marks) A causal digital filter with input $x[n]$ and output $y[n]$ is governed by the relationship

$$y[n] = x[n] + x[n-2] + y[n-1] - 0.5y[n-2].$$

(a) Show that the system function can be written as

$$H(z) = \frac{z^2+1}{(z-z_0)(z-z_0^*)}$$

where $z_0 = (1+j)/2$ and z_0^* is the complex conjugate of z_0.

(b) Sketch the poles and zeros of this filter in the z-plane.

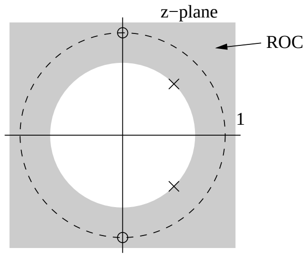
(c) Determine an expression for the impulse response of the filter. You may write your solution in terms of undetermined coefficients along with a set of simultaneous equations that specify them.

(d) Is the filter stable?

(a) Since $Y(z) = X(z) + z^{-2}X(z) + z^{-1}Y(z) - 0.5z^{-2}Y(z)$ the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1-z^{-1}+0.5z^{-2}} = \frac{z^2+1}{z^2-z+0.5} = \frac{z^2+1}{(z-z_0)(z-z_0^*)}.$$

(b) Poles at $z = z_0$ and $z = z_0^*$. Zeros where $z^2 = -1$, or at $z = \pm j$.



(c) Using partial fractions the system function can be written as

$$H(z) = \frac{1+z^{-2}}{(1-z_0z^{-1})(1-z_0^*z^{-1})} = A + \frac{B}{(1-z_0z^{-1})} + \frac{B}{(1-z_0^*z^{-1})}.$$

For a causal ROC we have $|z| > |z_0|$ and the inverse is

$$h[n] = A\delta[n] + B(z_0)^n u[n] + C(z_0^*)^n u[n],$$

where $A = 2$, $A + B + C = 1$, and $A + Bz_0^* + Cz_0 = 0$.

(d) The system is stable because the ROC includes the unit circle.

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n-n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$