EEE4001F: Digital Signal Processing

Class Test 1

16 March 2012

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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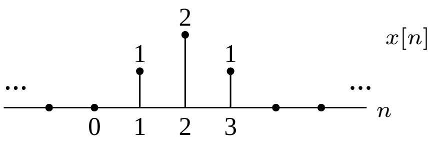
1. (5 marks) For the signal $x[n]$ below



plot the following:

(a) $y_1[n] = x[n] + x[-n]$

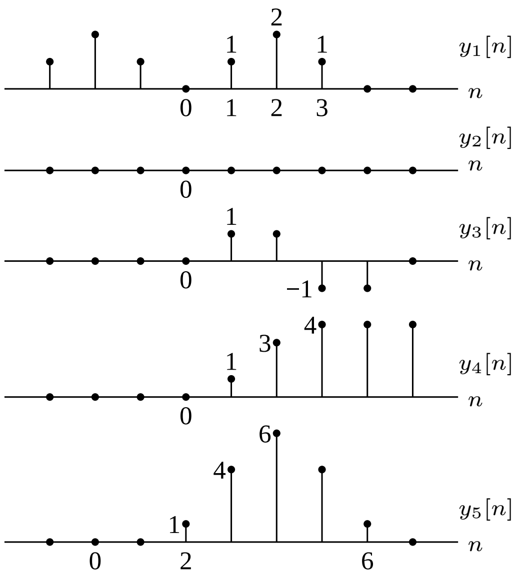
(b) $y_2[n] = x[n]x[-n]$

(c) $y_3[n] = x[n] - x[n-1]$

(d) $y_4[n] = \sum_{k=-\infty}^{n} x[k]$

(e) $y_5[n] = x[n] * x[n]$.

Plots follow:



2. (5 marks) For what values of ω is the signal $x[n] = e^{j\omega n}$ periodic with a period of 8?

For $x[n]$ to be periodic with period N we require that $x[n] = x[n+N]$ for all n. For the given $x[n]$ this requires that

$$e^{j\omega n} = e^{j\omega(n+N)} = e^{j\omega N} e^{j\omega n}$$

for all n, which means that $e^{j\omega N} = 1 = e^{j2\pi k}$ for any integer k. Thus ω must satisfy

$$\omega = \frac{2\pi k}{N} = \frac{2\pi}{8} k = \frac{\pi}{4} k$$

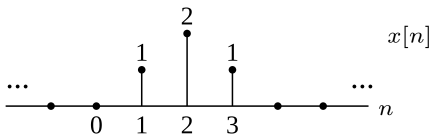
for any integer k. The set of frequencies ω for which the signal is periodic with a period of 8 is therefore

$$\omega \in \left\{ \ldots, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \ldots \right\}.$$

3. (5 marks) Suppose $x[n]$ has the DTFT $X(e^{j\omega})$, and consider the signal $y[n]$ defined as follows:

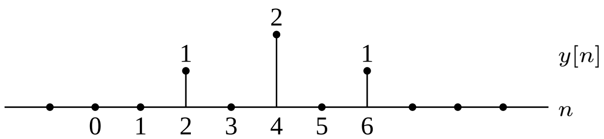
$$y[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

(a) Plot $y[n]$ if $x[n]$ is the following:



(b) Find a general expression for the DTFT $y(e^{j\omega})$ of $y[n]$ in terms of $X(e^{j\omega})$.

(a) The signal is expanded on the n axis as follows:



(b) Since

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\ &= \cdots + y[-3]e^{j\omega 3} + y[-2]e^{j\omega 2} + y[-1]e^{j\omega 1} + y[0]e^{j\omega 0} + y[1]e^{-j\omega 1} \\ &\qquad + y[2]e^{-j\omega 2} + y[3]e^{-j\omega 3} + \cdots \\ &= \cdots + 0e^{j\omega 3} + x[-1]e^{j\omega 2} + 0e^{j\omega 1} + x[0]e^{j\omega 0} + 0e^{-j\omega 1} \\ &\qquad + x[1]e^{-j\omega 2} + 0e^{-j\omega 3} + \cdots \\ &= \cdots + x[-1]e^{j\omega 2} + x[0]e^{j\omega 0} + x[1]e^{-j\omega 2} + \cdots = \sum_{n=-\infty}^{\infty} x[n]e^{-j(2\omega)n} \end{aligned}$$

we evidently have $Y(e^{j\omega}) = X(e^{j(2\omega)})$, which is a compression of the signal on the ω axis.

4. (5 marks) A linear time-invariant system has system function

$$H(z) = \frac{6z-2}{6z-3}, \qquad |z| > 1/2.$$

Find the impulse response of a stable inverse system. Is this inverse system causal?

The system function can be written as

$$H(z) = \frac{z-1/3}{z-1/2} = \frac{1-1/3z^{-1}}{1-1/2z^{-1}}$$

so the inverse is

$$G(z) = \frac{z-1/2}{z-1/3} = \frac{1-1/2z^{-1}}{1-1/3z^{-1}}.$$

This has a pole at $z = 1/3$, so the two possible ROCs are $|z| < 1/3$ and $|z| > 1/3$. If $|z| > 1/3$ then the inverse system is stable (ROC contains the unit circle) and it will also be causal (ROC extends out to infinity).

The inverse in this case is the right-sided sequence

$$h[n] = (1/3)^n u[n] - \frac{1}{2}(1/3)^{n-1} u[n-1].$$

Fourier transform properties

Sequences $x[n], y[n]$	Transforms $X(e^{j\omega}), Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$