EEE4001F: Digital Signal Processing

Class Test 2

21 April 2011

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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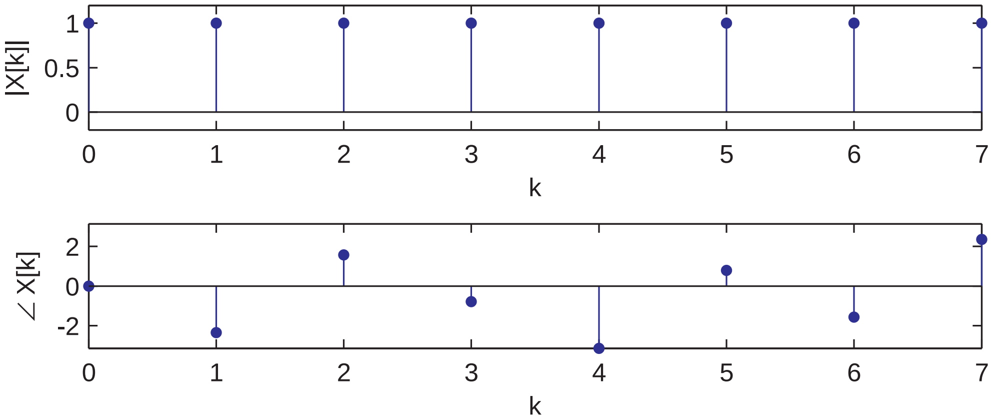
1. (5 marks) Determine the 8-point DFT of the real-valued sequence $x[n] = \delta[n-3]$. Plot the magnitude and phase of your answer on separate axes, ensuring that the phase lies between $-\pi$ and π.

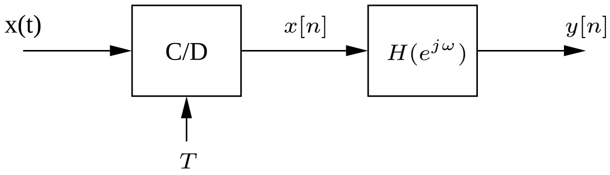
Since

$$X[k] = \sum_{n=0}^{7} \delta[n-3] e^{-j2\pi kn/8} = \sum_{n=0}^{7} \delta[n-3] e^{-j2\pi k3/8} = e^{-j\pi k3/4},$$

we have $|X[k]| = 1$ and $\angle X[k] = -\frac{3}{4}\pi k$. Plots as follows:



2. (5 marks) Consider the system below



where $T = 0.001$s and

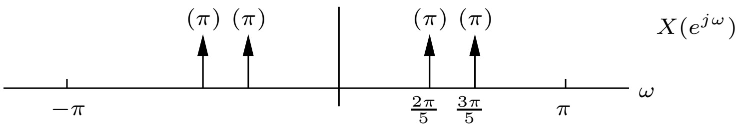
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq 0.5\pi \\ 0 & \text{otherwise} \end{cases}$$

for $-\pi \leq \omega \leq \pi$. Find the output $y[n]$ if the input is $x(t) = \cos(400\pi t) + \cos(600\pi t)$.

The discretised input is

$$x[n] = x(nT) = \cos\left(\frac{400}{1000}\pi n\right) + \cos\left(\frac{600}{1000}\pi n\right) = \cos\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{3\pi}{5}n\right)$$
$$= \frac{1}{2}e^{j\frac{2\pi}{5}n} + \frac{1}{2}e^{-j\frac{2\pi}{5}n}\frac{1}{2}e^{j\frac{3\pi}{5}n} + \frac{1}{2}e^{-j\frac{3\pi}{5}n}.$$

In the frequency domain this is



and the filter removes the two impulses at frequencies $\omega = \pm\frac{3\pi}{5}$, and hence the $\cos\left(\frac{3\pi}{5}n\right)$ term. The output is therefore just the remaining term

$$y[n] = \cos\left(\frac{2\pi}{5}n\right).$$

3. (5 marks) Find $w[n] = x[n] * y[n]$ with

$$x[n] = e^{j\pi n/3} \quad \text{and} \quad y[n] = \frac{\sin(\pi(n-5)/2)}{\pi(n-5)}.$$

The signal $y[n] = y_0[n-5]$, where $y_0[n] = \frac{\sin(\pi n/2)}{\pi n}$. From the Fourier tables we see that

$$y_0[n] = \frac{\sin(\pi n/2)}{\pi n} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad Y_0(e^{j\omega}) = \begin{cases} 1 & |\omega| < \pi/2 \\ 0 & \pi/2 < |\omega| < \pi. \end{cases}$$

Applying the time shift property gives the pair

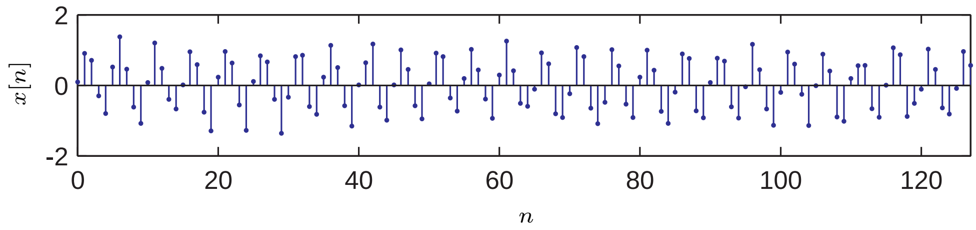
$$y[n] = \frac{\sin(\pi(n-5)/2)}{\pi(n-5)} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad Y(e^{j\omega}) = e^{-j\omega 5} Y_0(e^{j\omega}).$$

We can think of $w[n]$ as the output of a system with impulse response $y[n]$, driven by the input $x[n]$. Since $x[n]$ is a complex exponential with frequency $\omega = \pi/3$ the output will be

$$w[n] = Y(e^{j\pi/3})e^{j\pi n/3} = e^{-j5\pi/3}Y(e^{j\pi/3})e^{j\pi n/3} = e^{j(\pi n/3 - 5\pi/3)},$$

since $Y(e^{j\pi/3}) = 1$.

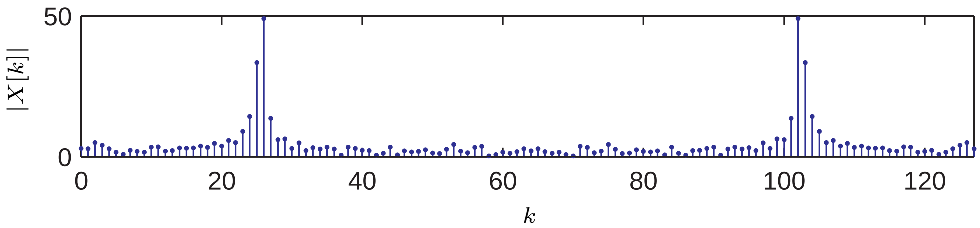
4. (5 marks) Two students want to analyse a signal from a microphone. They digitise a sample of 128 points, obtaining the signal below:



They decide to investigate the apparent periodicity by looking at the signal in the frequency domain. To do this they calculate the DFT

$$X[k] = \sum_{n=0}^{127} x[n] e^{-j \frac{2\pi}{128} kn}$$

for $k = 0, \ldots, 127$, which has the following magnitude plot:



(a) What is the dominant frequency present in the signal, measured in radians per sample?

(b) The quantity $|X[k]|$ as calculated is a poor estimate of the spectrum of the microphone signal. Why is this so? What can be done to improve the estimate?

(a) The leftmost peak of the frequency response occurs at about $k = 26$. This corresponds to the discrete complex exponential $e^{j \frac{2\pi}{128}(26)n}$, which has a frequency $\omega = \frac{2\pi}{128}(26)$ radians per sample. The rightmost peak just corresponds to the same frequency, but with opposite sign (the signal $x[n]$ is real, so the transform is symmetric).

(b) Since the sum in the DFT is only over 128 samples, a rectangular window $w_r[n]$ of this length has effectively been applied to the signal. We are therefore observing samples of the DTFT of the windowed signal $x[n]w_r[n]$. The samples of $X[k]$ therefore relate to the spectrum of $X(e^{j\omega}) * W_r(e^{j\omega})$, so we are seeing the desired spectrum "blurred" by the window function $W_r(e^{j\omega})$, which is of a sinc form.

The peaks in $|X[k]|$ are broad because the sinc function has high sidelobes. These can be reduced by using a different window function that has better spectral properties. The estimate can also be improved by capturing a longer sample of data: this increases the length of the window function $w[n]$ in the time domain, which compresses the corresponding $W(e^{j\omega})$ in frequency. Ideally $W(e^{j\omega})$ should be a Dirac delta function, so this is a good thing.

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$