EEE4001F: Digital Signal Processing

Class Test 1

11 March 2010

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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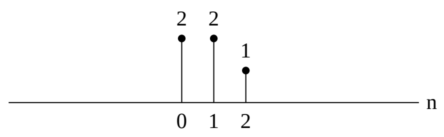
1. (5 marks) If $x[n]$ is the signal



then sketch the following:

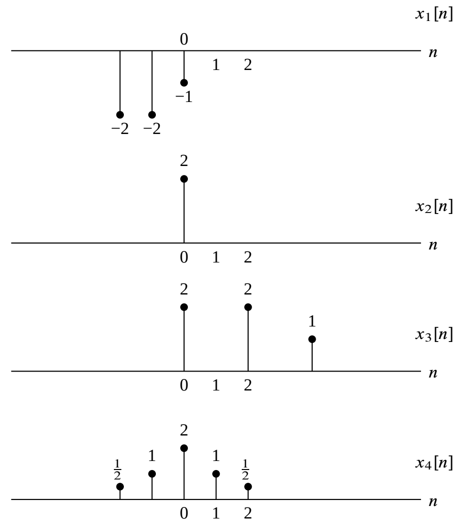
(a) $x_1[n] = -x[n+2]$

(b) $x_2[n] = x[2n+1]$

(c) $x_3[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

(d) $x_4[n] = \frac{1}{2}(x[n] + x[-n])$.

Plots are as follows:



2. (5 marks) A linear system has the relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k]$$

between its input $x[n]$ and its output $y[n]$, where $g[n] = u[n] - u[n-4]$.

(a) Determine $y[n]$ when $x[n] = \delta[n-1]$.

(b) Determine $y[n]$ when $x[n] = \delta[n-2]$.

(c) Is the system LTI?

(a) Output for $x[n] = \delta[n-1]$ is

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[k-1]g[n-2k] = \sum_{k=-\infty}^{\infty} \delta[k-1]g[n-2(1)]$$
$$= g[n-2] \sum_{k=-\infty}^{\infty} \delta[k-1] = g[n-2].$$

(b) Output for $x[n] = \delta[n-2]$ is

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[k-2]g[n-2k] = \sum_{k=-\infty}^{\infty} \delta[k-2]g[n-2(2)] = g[n-4].$$

(c) From part (a) we have the following input-output pair:

$$x[n] = \delta[n-1] \longrightarrow g[n-2] = y[n].$$

If the system is time invariant then $x[n-1] \longrightarrow y[n-1]$, so we must have $\delta[n-2] \longrightarrow g[n-3]$. But from (b) we see that $\delta[n-2] \longrightarrow g[n-4]$, so time invariance does not hold, and the system is not LTI.

3. (5 marks) Consider two systems described by the following linear constant coefficient difference equations:

$$y[n] = 0.2y[n-1] + x[n] + 0.3x[n-1] + 0.02x[n-2]$$
$$y[n] = x[n] - 0.1x[n-1].$$

Prove that the two systems are equivalent.

Using the Z-transform the first LCCDE can be written as

$$Y(z) = 0.2z^{-1}Y(z) + X(z) + 0.3z^{-1}X(z) + 0.02z^{-2}X(z)$$

so the system function is

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.3z^{-1} + 0.02z^{-2}}{1 - 0.2z^{-1}} = \frac{(1 - 0.2z^{-1})(1 - 0.1z^{-1})}{1 - 0.2z^{-1}}$$
$$= 1 - 0.1z^{-1} = \frac{z - 0.1}{z}.$$

Since there is only one pole at the origin, the ROC is all z (excluding $z = 0$).

The second LCCDE has an identical system function, so the LCCDEs must represent the same system.

4. (5 marks) The input $x[n]$ and the output $y[n]$ of a causal system obeys the relationship

$$y[n] = x[n] + 0.3y[n-2].$$

Find the impulse response, and determine whether the system is BIBO stable or not.

The Z-transform of the difference equation is

$$Y(z) = X(z) + 0.3z^{-2}Y(z),$$

so the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.3z^{-2}} = \frac{z^2}{z^2-0.3} = \frac{z^2}{(z-\sqrt{0.3})(z+\sqrt{0.3})}$$

$$= \frac{1}{(1-\sqrt{0.3}z-1)(1+\sqrt{0.3}z^{-1})} = \frac{a}{1-\sqrt{0.3}z-1} + \frac{b}{1+\sqrt{0.3}z-1}.$$

This has two poles at $z = \pm\sqrt{0.3}$, so for a causal system we must have ROC $|z| > \sqrt{0.3}$. This ROC includes the unit circle, so the system is stable. Inverting gives the impulse response

$$h[n] = a(-\sqrt{0.3})^n u[n] + b(\sqrt{0.3})^n u[n].$$

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n-n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$