Discrete-time signals and systems

See Oppenheim and Schafer, Second Edition pages 8–93, or First Edition pages 8–79.

1 Discrete-time signals

A discrete-time signal is represented as a sequence of numbers:

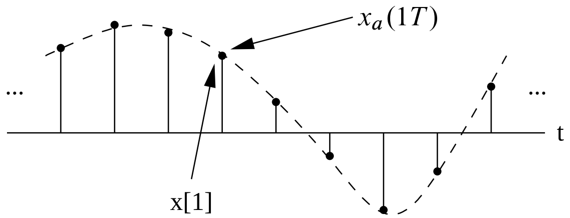
$$x = \{x[n]\}, \qquad -\infty < n < \infty.$$

Here n is an integer, and $x[n]$ is the nth sample in the sequence.

Discrete-time signals are often obtained by sampling continuous-time signals. In this case the nth sample of the sequence is equal to the value of the analogue signal $x_a(t)$ at time $t = nT$:

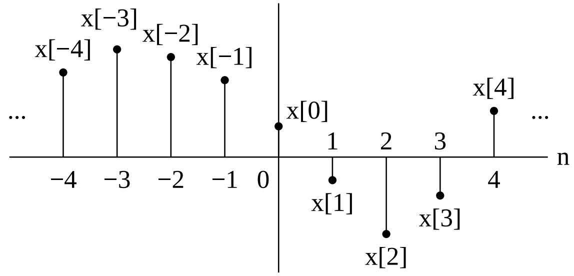
$$x[n] = x_a(nT), \qquad -\infty < n < \infty.$$

The **sampling period** is then equal to T, and the sampling frequency is $f_s = 1/T$.



For this reason, although $x[n]$ is strictly the nth number in the sequence, we often refer to it as the nth **sample**. We also often refer to "the sequence $x[n]$" when we mean the entire sequence.

Discrete-time signals are often depicted graphically as follows:



(This can be plotted using the MATLAB function `stem`.) The value $x[n]$ is **undefined** for noninteger values of n.

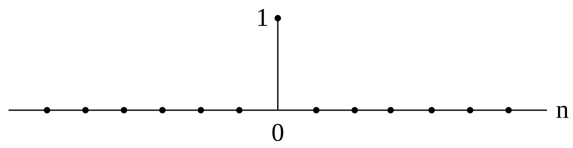
Sequences can be manipulated in several ways. The **sum** and **product** of two sequences $x[n]$ and $y[n]$ are defined as the sample-by-sample sum and product respectively. Multiplication of $x[n]$ by a is defined as the multiplication of each sample value by a.

A sequence $y[n]$ is a **delayed** or **shifted** version of $x[n]$ if

$$y[n] = x[n - n_0],$$

with n_0 an integer.

The **unit sample sequence**

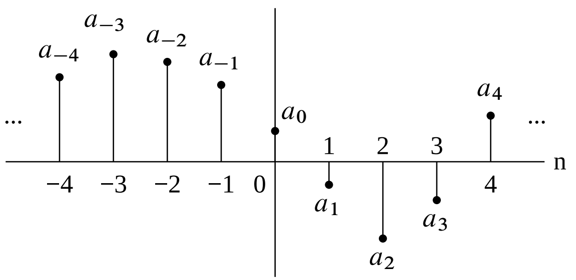


is defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0. \end{cases}$$

This sequence is often referred to as a **discrete-time impulse**, or just **impulse**. It plays the same role for discrete-time signals as the Dirac delta function does for continuous-time signals. However, there are no mathematical complications in its definition.

An important aspect of the impulse sequence is that an arbitrary sequence can be represented as a sum of scaled, delayed impulses. For example, the sequence



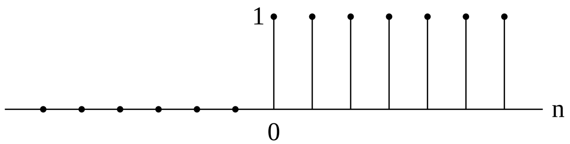
can be represented as

$$x[n] = a_{-4}\delta[n+4] + a_{-3}\delta[n+3] + a_{-2}\delta[n+2] + a_{-1}\delta[n+1] + a_0\delta[n] \\ + a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4].$$

In general, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

The **unit step sequence**



is defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

The unit step is related to the impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k].$$

Alternatively, this can be expressed as

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots = \sum_{k=0}^{\infty} \delta[n-k].$$

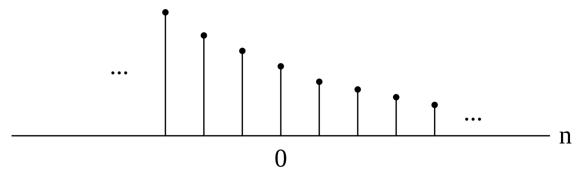
Conversely, the unit sample sequence can be expressed as the first backward difference of the unit step sequence

$$\delta[n] = u[n] - u[n-1].$$

Exponential sequences are important for analysing and representing discrete-time systems. The general form is

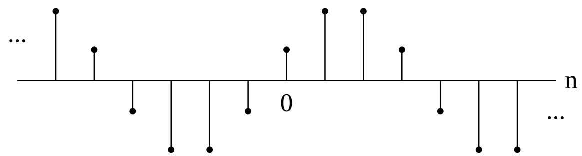
$$x[n] = A\alpha^n.$$

If A and α are real numbers then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n:



For $-1 < \alpha < 0$ the sequence alternates in sign, but decreases in magnitude. For $|\alpha| > 1$ the sequence grows in magnitude as n increases.

A **sinusoidal sequence**



has the form

$$x[n] = A\cos(\omega_0 n + \phi) \qquad \text{for all } n,$$

with A and ϕ real constants. The exponential sequence $A\alpha^n$ with complex $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$ can be expressed as

$$\begin{aligned} x[n] = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} = |A||\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi), \end{aligned}$$

so the real and imaginary parts are exponentially weighted sinusoids.

When $|\alpha| = 1$ the sequence is called the **complex exponential sequence**:

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi).$$

The **frequency** of this complex sinusoid is ω_0, and is measured in radians per sample. The **phase** of the signal is ϕ.

The index n is always an integer. This leads to some important differences between the properties of discrete-time and continuous-time complex exponentials:

- Consider the complex exponential with frequency $(\omega_0 + 2\pi)$:

  $$x[n] = Ae^{j(\omega_0+2\pi)n} = Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}.$$

  Thus the sequence for the complex exponential with frequency ω_0 is *exactly* the same as that for the complex exponential with frequency $(\omega_0 + 2\pi)$. More generally, complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another. Similarly, for sinusoidal sequences

  $$x[n] = A\cos[(\omega_0 + 2\pi r)n + \phi] = A\cos(\omega_0 n + \phi).$$

- In the continuous-time case, sinusoidal and complex exponential sequences are always periodic. Discrete-time sequences are periodic (with period N) if

  $$x[n] = x[n + N] \qquad \text{for all } n.$$

  Thus the discrete-time sinusoid is only periodic if

  $$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi),$$

  which requires that

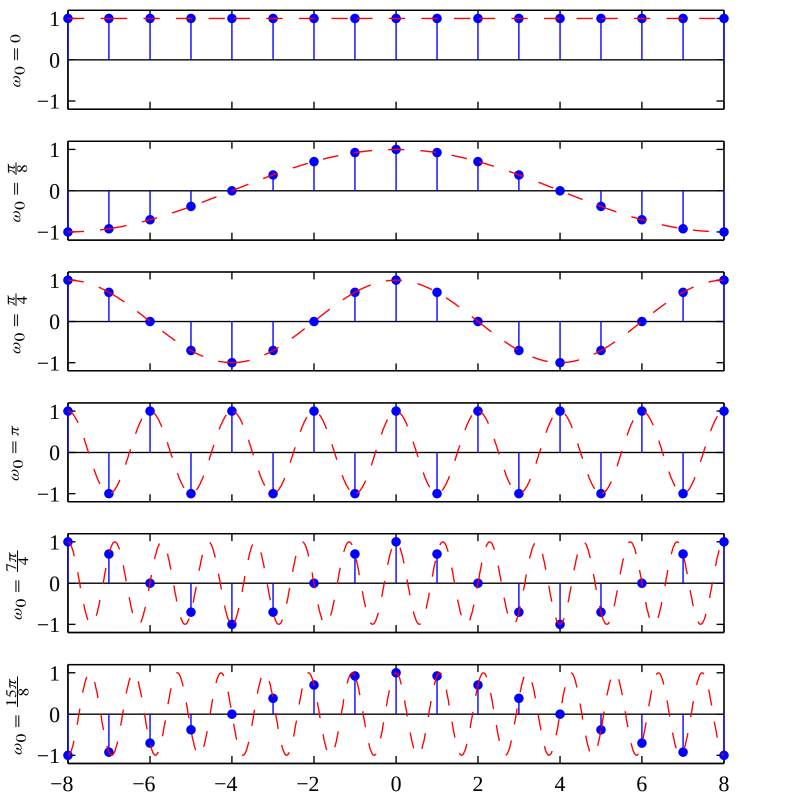
  $$\omega_0 N = 2\pi k \qquad \text{for } k \text{ an integer.}$$

  The same condition is required for the complex exponential sequence $Ce^{j\omega_0 n}$ to be periodic.

The two factors just described can be combined to reach the conclusion that there are only N distinguishable frequencies for which the corresponding sequences are periodic with period N. One such set is

$$\omega_k = \frac{2\pi k}{N}, \qquad k = 0, 1, \ldots, N-1.$$

Additionally, for discrete-time sequences the interpretation of high and low frequencies has to be modified: the discrete-time sinusoidal sequence $x[n] = A\cos(\omega_0 n + \phi)$ oscillates more rapidly as ω_0 increases from 0 to π, but the oscillations become slower as it increases further from π to 2π.

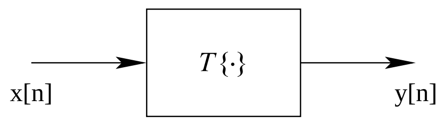


The sequence corresponding to $\omega_0 = 0$ is indistinguishable from that with $\omega_0 = 2\pi$. In general, any frequencies in the vicinity of $\omega_0 = 2\pi k$ for integer k are typically referred to as low frequencies, and those in the vicinity of $\omega_0 = (\pi + 2\pi k)$ are high frequencies.

2 Discrete-time systems

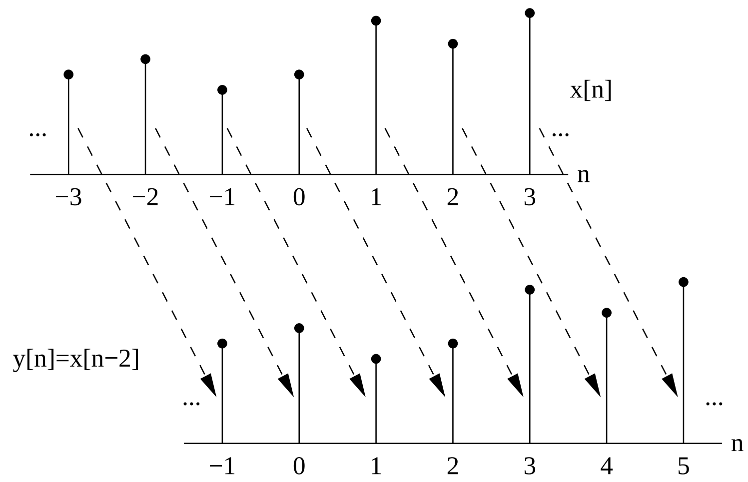
A discrete-time system is defined as a transformation or mapping operator that maps an input signal $x[n]$ to an output signal $y[n]$. This can be denoted as

$$y[n] = T\{x[n]\}.$$



Example: Ideal delay

$$y[n] = x[n - n_d]:$$

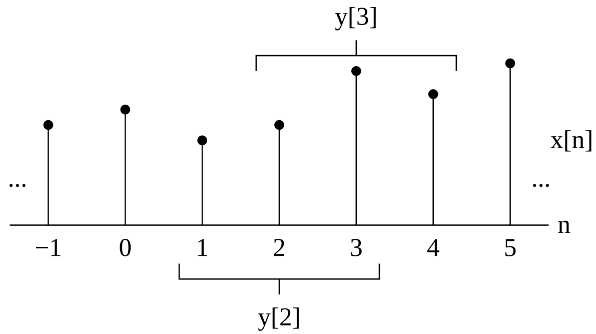


This operation shifts input sequence later by n_d samples.

Example: Moving average

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

For $M_1 = 1$ and $M_2 = 1$, the input sequence



yields an output with

$$\begin{aligned} &\vdots \\ y[2] &= \frac{1}{3}(x[1] + x[2] + x[3]) \\ y[3] &= \frac{1}{3}(x[2] + x[3] + x[4]) \\ &\vdots \end{aligned}$$

In general, systems can be classified by placing constraints on the transformation $T\{\cdot\}$.

2.1 Memoryless systems

A system is memoryless if the output $y[n]$ depends only on $x[n]$ at the same n. For example, $y[n] = (x[n])^2$ is memoryless, but the ideal delay

$y[n] = x[n - n_d]$ is not unless $n_d = 0$.

2.2 Linear systems

A system is linear if the principle of superposition applies. Thus if $y_1[n]$ is the response of the system to the input $x_1[n]$, and $y_2[n]$ the response to $x_2[n]$, then linearity implies

- **Additivity:**

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

- **Scaling:**

$$T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n].$$

These properties combine to form the general principle of superposition

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} = ay_1[n] + by_2[n].$$

In all cases a and b are arbitrary constants.

This property generalises to many inputs, so the response of a linear system to $x[n] = \sum_k a_k x_k[n]$ will be $y[n] = \sum_k a_k y_k[n]$.

2.3 Time-invariant systems

A system is time invariant if a time shift or delay of the input sequence causes a corresponding shift in the output sequence. That is, if $y[n]$ is the response to $x[n]$, then $y[n - n_0]$ is the response to $x[n - n_0]$.

For example, the accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is time invariant, but the compressor system

$$y[n] = x[Mn]$$

for M a positive integer (which selects every Mth sample from a sequence) is not.

2.4 Causality

A system is causal if the output at n depends only on the input *at* n *and earlier inputs*.

For example, the backward difference system

$$y[n] = x[n] - x[n-1]$$

is causal, but the forward difference system

$$y[n] = x[n+1] - x[n]$$

is not.

2.5 Stability

A system is stable if every bounded input sequence produces a bounded output sequence:

- **Bounded input:** $|x[n]| \leq B_x < \infty$
- **Bounded output:** $|y[n]| \leq B_y < \infty$.

For example, the accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[n]$$

is an example of an *unbounded* system, since its response to the unit step $u[n]$ is

$$y[n] = \sum_{k=-\infty}^{n} u[n] = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0, \end{cases}$$

which has no finite upper bound.

3 Linear time-invariant systems

If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses, then it follows that a linear time-invariant (LTI) system can be completely characterised by its impulse response.

Suppose $h_k[n]$ is the response of a linear system to the impulse $\delta[n-k]$ at $n = k$. Since

$$y[n] = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\},$$

the principle of superposition means that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k] h_k[n].$$

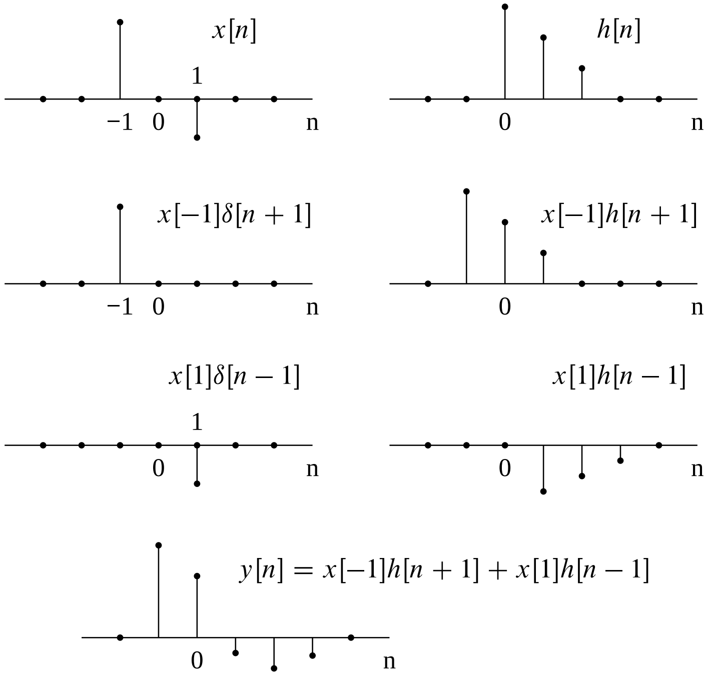
If the system is additionally time invariant, then the response to $\delta[n-k]$ is $h[n-k]$. The previous equation then becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k].$$

This expression is called the **convolution sum**. Therefore, a LTI system has the property that given $h[n]$, we can find $y[n]$ for *any* input $x[n]$. Alternatively, $y[n]$ is the **convolution** of $x[n]$ with $h[n]$, denoted as follows:

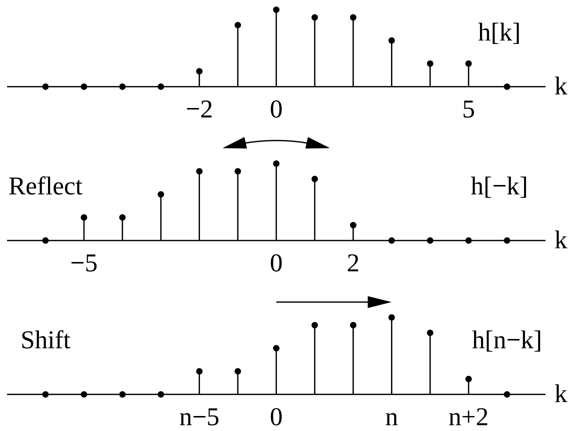
$$y[n] = x[n] * h[n].$$

The previous derivation suggests the interpretation that the input sample at $n = k$, represented by $x[k]\delta[n-k]$, is transformed by the system into an output sequence $x[k]h[n-k]$. For each k, these sequences are superimposed to yield the overall output sequence:



A slightly different interpretation, however, leads to a convenient computational form: the nth value of the output, namely $y[n]$, is obtained by multiplying the input sequence (expressed as a function of k) by the sequence with values $h[n-k]$, and then summing all the values of the products $x[k]h[n-k]$. The key to this method is in understanding how to form the sequence $h[n-k]$ for all values of n of interest.

To this end, note that $h[n-k] = h[-(k-n)]$. The sequence $h[-k]$ is seen to be equivalent to the sequence $h[k]$ reflected around the origin:



The sequence $h[n-k]$ is then obtained by shifting the origin of the sequence to $k = n$.

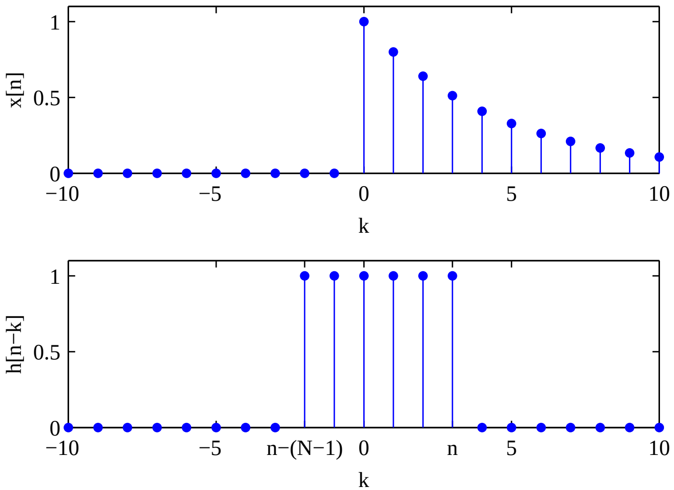
To implement discrete-time convolution, the sequences $x[k]$ and $h[n-k]$ are multiplied together for $-\infty < k < \infty$, and the products summed to obtain the value of the output sample $y[n]$. To obtain another output sample, the procedure is repeated with the origin shifted to the new sample position.

Example: analytical evaluation of the convolution sum

Consider the output of a system with impulse response

$$h[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

to the input $x[n] = a^n u[n]$. To find the output at n, we must form the sum over all k of the product $x[k]h[n-k]$.



Since the sequences are non-overlapping for all negative n, the output must be zero:

$$y[n] = 0, \qquad n < 0.$$

For $0 \le n \le N-1$ the product terms in the sum are $x[k]h[n-k] = a^k$, so it follows that

$$y[n] = \sum_{k=0}^{n} a^k, \qquad 0 \le n \le N-1.$$

Finally, for $n > N-1$ the product terms are $x[k]h[n-k] = a^k$ as before, but the lower limit on the sum is now $n - N + 1$. Therefore

$$y[n] = \sum_{k=n-N+1}^{n} a^k, \qquad n > N-1.$$

4 Properties of LTI systems

All LTI systems are described by the convolution sum

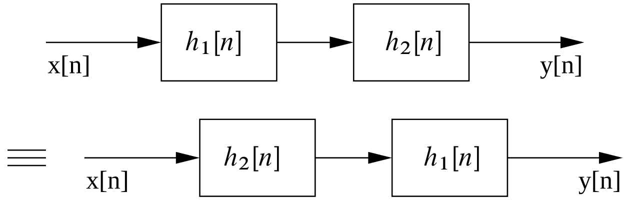
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Some properties of LTI systems can therefore be found by considering the properties of the convolution operation:

- **Commutative:** $x[n] * h[n] = h[n] * x[n]$
- **Distributive over addition:**

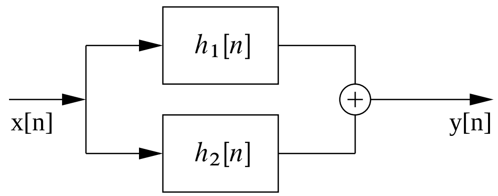
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

- **Cascade connection:**



$$y[n] = h[n] * x[n] = h_1[n] * h_2[n] * x[n] = h_2[n] * h_1[n] * x[n].$$

- **Parallel connection:**



$$y[n] = (h_1[n] + h_2[n]) * x[n] = h_p[n] * x[n].$$

Additional important properties are:

- A LTI system is **stable** if and only if $S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$. The **ideal**

delay system $h[n] = \delta[n - n_d]$ is stable since $S = 1 < \infty$; the **moving average** system

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$
$$= \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise,} \end{cases}$$

the **forward difference** system $h[n] = \delta[n+1] - \delta[n]$, and the **backward difference** system $h[n] = \delta[n] - \delta[n-1]$ are stable since S is the sum of a finite number of finite samples, and is therefore less than ∞; the **accumulator** system

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$
$$= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$
$$= u[n]$$

is unstable since $S = \sum_{n=0}^{\infty} u[n] = \infty$.

- A LTI system is causal if and only if $h[n] = 0$ for $n < 0$. The ideal delay system is causal if $n_d \geq 0$; the moving average system is causal if $-M_1 \geq 0$ and $M_2 \geq 0$; the accumulator and backward difference systems are causal; the forward difference system is noncausal.

Systems with only a finite number of nonzero values in $h[n]$ are called **Finite duration impulse response (FIR)** systems. FIR systems are stable if each impulse response value is finite. The ideal delay, the moving average, and the forward and backward difference described above fall into this class. **Infinite impulse response (IIR)** systems, such as the accumulator system, are more difficult to analyse. For example, the accumulator system is unstable, but the

IIR system

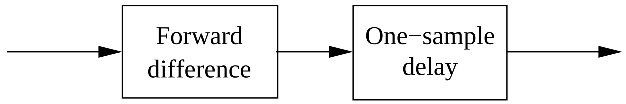
$$h[n] = a^n u[n], \qquad |a| < 1$$

is stable since

$$S = \sum_{n=0}^{\infty} |a^n| \leq \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$$

(it is the sum of an infinite geometric series).

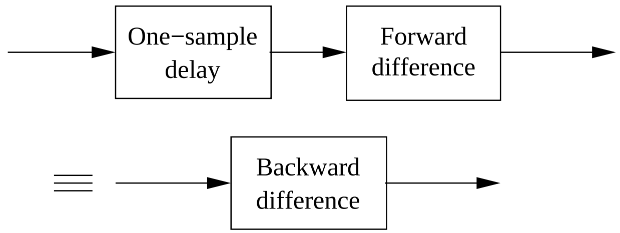
Consider the system



which has

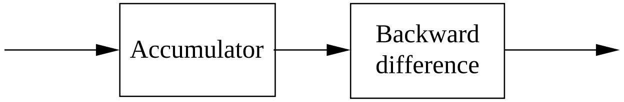
$$h[n] = (\delta[n+1] - \delta[n]) * \delta[n-1]$$
$$= \delta[n-1] * \delta[n+1] - \delta[n-1] * \delta[n]$$
$$= \delta[n] - \delta[n-1].$$

This is the impulse response of a backward difference system:



Here a non-causal system has been converted to a causal one by cascading with a delay. In general, *any non-causal FIR system can be made causal by cascading with a sufficiently long delay*.

Consider the system consisting of an accumulator followed by a backward difference:



The impulse response of this system is

$$h[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n].$$

The output is therefore equal to the input because $x[n] * \delta[n] = x[n]$. Thus the backward difference exactly compensates for (or inverts) the effect of the accumulator — the backward difference system is the **inverse system** for the accumulator, and vice versa. We define this inverse relationship for all LTI systems:

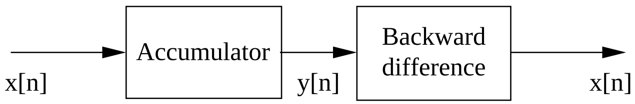
$$h[n] * h_i[n] = \delta[n].$$

5 Linear constant coefficient difference equations

Some LTI systems can be represented in terms of linear constant coefficient difference (LCCD) equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

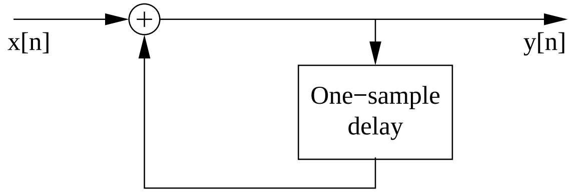
Example: difference equation representation of the accumulator
Take for example the accumulator



Here $y[n] - y[n-1] = x[n]$, which can be written in the desired form with $N = 1$, $a_0 = 1$, $a_1 = -1$, $M = 0$, and $b_0 = 1$. Rewriting as

$$y[n] = y[n-1] + x[n]$$

we obtain the **recursion representation**



where at n we add the current input $x[n]$ to the previously accumulated sum $y[n-1]$.

Example: difference equation representation of moving average
Consider now the moving average system with $M_1 = 0$:

$$h[n] = \frac{1}{M_2 + 1}(u[n] - u[n - M_2 - 1]).$$

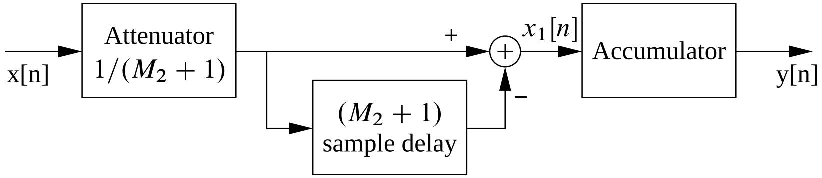
The output of the system is

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k],$$

which is a LCCDE with $N = 0$, $a_0 = 1$, and $M = M_2$, $b_k = 1/(M_2 + 1)$. Using the sifting property of $\delta[n]$,

$$h[n] = \frac{1}{M_2 + 1}(\delta[n] - \delta[n - M_2 - 1]) * u[n]$$

so



Here $x_1[n] = 1/(M_2+1)(x[n] - x[n - M_2 - 1])$ and for the accumulator $y[n] - y[n-1] = x_1[n]$. Therefore

$$y[n] - y[n-1] = \frac{1}{M_2+1}(x[n] - x[n - M_2 - 1]),$$

which is again a (different) LCCD equation with $N = 1$, $a_0 = 1$, $a_1 = -1$, $b_0 = -b_{M_2+1} = 1/(M_2+1)$.

As for constant coefficient differential equations in the continuous case, without additional information or constraints a LCCDE does not provide a unique solution for the output given an input. Specifically, suppose we have the particular output $y_p[n]$ for the input $x_p[n]$. The same equation then has the solution

$$y[n] = y_p[n] + y_h[n],$$

where $y_h[n]$ is any solution with $x[n] = 0$. That is, $y_h[n]$ is an **homogeneous solution** to the **homogeneous equation**

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0.$$

It can be shown that there are N nonzero solutions to this equation, so a set of N auxiliary conditions are required for a unique specification of $y[n]$ for a given $x[n]$.

If a system is LTI *and causal*, then the initial conditions are **initial rest** conditions, and a unique solution can be obtained.

6 Frequency-domain representation of discrete-time signals and systems

The Fourier transform considered here is strictly speaking the **discrete-time Fourier transform (DTFT)**, although Oppenheim and Schafer call it just the

Fourier transform. Its properties are recapped here (with examples) to show nomenclature.

Complex exponentials

$$x[n] = e^{j\omega n}, \qquad -\infty < n < \infty$$

are eigenfunctions of LTI systems:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}\right).$$

Defining

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

we have that $y[n] = H(e^{j\omega})e^{j\omega n} = H(e^{j\omega})x[n]$. Therefore, $e^{j\omega n}$ is an eigenfunction of the system, and $H(e^{j\omega})$ is the associated eigenvalue.

The quantity $H(e^{j\omega})$ is called the **frequency response** of the system, and

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) = |H(e^{j\omega})| e^{j\sphericalangle H(e^{j\omega})}.$$

Example: frequency response of ideal delay:
Consider the input $x[n] = e^{j\omega n}$ to the ideal delay system $y[n] = x[n - n_d]$: the output is

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}.$$

The frequency response is therefore

$$H(e^{j\omega}) = e^{-j\omega n_d}.$$

Alternatively, for the ideal delay $h[n] = \delta[n - n_d]$,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}.$$

The real and imaginary parts of the frequency response are

$H_R(e^{j\omega}) = \cos(\omega n_d)$ and $H_I(e^{j\omega}) = \sin(\omega n_d)$, or alternatively

$$|H(e^{j\omega})| = 1$$
$$\sphericalangle H(e^{j\omega}) = -\omega n_d.$$

The frequency response of a LTI system is essentially the same for continuous and discrete time systems. However, an important distinction is that in the discrete case it is *always* periodic in frequency with a period 2π:

$$\begin{aligned} H(e^{j(\omega+2\pi)}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}e^{-j2\pi n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = H(e^{j\omega}). \end{aligned}$$

This last result holds since $e^{\pm j2\pi n} = 1$ for integer n.

The reason for this periodicity is related to the observation that the sequence

$$\left\{e^{-j\omega n}\right\}, \qquad -\infty < n < \infty$$

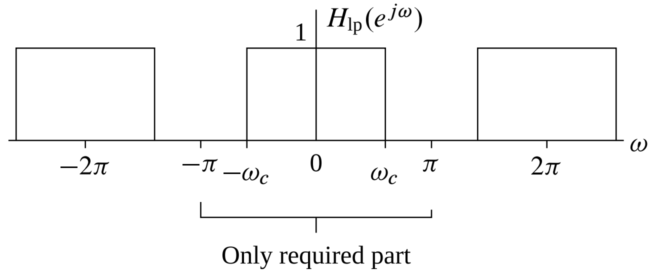
has exactly the same values as the sequence

$$\left\{e^{-j(\omega+2\pi)n}\right\}, \qquad -\infty < n < \infty.$$

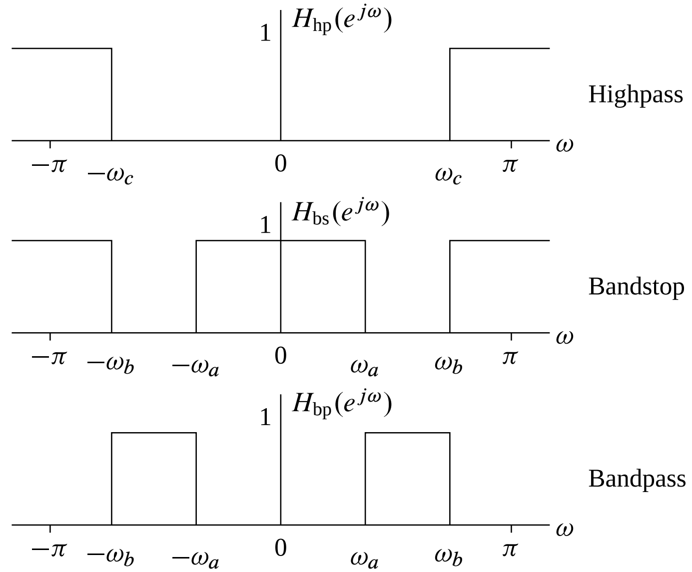
A system will therefore respond in exactly the same way to both sequences.

Example: ideal frequency selective filters

The frequency response of an ideal lowpass filter is as follows:



Due to the periodicity in the response, it is only necessary to consider one frequency cycle, usually chosen to be the range $-\pi$ to π. Other examples of ideal filters are:



In these cases it is implied that the frequency response repeats with period 2π outside of the plotted interval.

Example: frequency response of the moving-average system

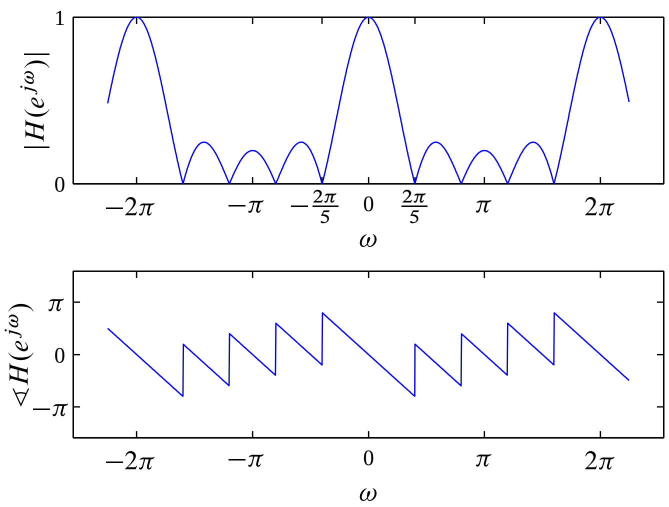
The frequency response of the moving average system

$$h[n] = \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \le n \le M_2 \\ 0 & \text{otherwise} \end{cases}$$

is given by

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1+M_2+1} \frac{e^{j\omega(M_2+M_1+1)/2} - e^{-j\omega(M_2+M_1+1)/2}}{1-e^{-j\omega}} e^{-\frac{j\omega(M_2-M_1+1)}{2}} \\ &= \frac{1}{M_1+M_2+1} \frac{e^{j\omega(M_2+M_1+1)/2} - e^{-j\omega(M_2+M_1+1)/2}}{e^{j\omega/2}-e^{-j\omega/2}} e^{-\frac{j\omega(M_2-M_1)}{2}} \\ &= \frac{1}{M_1+M_2+1} \frac{\sin[\omega(M_1+M_2+1)/2]}{\sin(\omega/2)} e^{-\frac{j\omega(M_2-M_1)}{2}}. \end{aligned}$$

For $M_1 = 0$ and $M_2 = 4$,



This system attenuates high frequencies (at around $\omega = \pi$), and therefore has the behaviour of a lowpass filter.

7 Fourier transforms of discrete sequences

The discrete time Fourier transform (DTFT) of the sequence $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

This is also called the **forward transform** or **analysis** equation. The **inverse Fourier transform**, or **synthesis** formula, is given by the Fourier integral

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega.$$

The Fourier transform is generally a complex-valued function of ω:

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}.$$

The quantities $|X(e^{j\omega})|$ and $\sphericalangle X(e^{j\omega})$ are referred to as the **magnitude** and **phase** of the Fourier transform. The Fourier transform is often referred to as the **Fourier spectrum**.

Since the frequency response of a LTI system is given by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k},$$

it is clear that the frequency response is equivalent to the Fourier transform of the impulse response, and the impulse response is

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n} d\omega.$$

A sufficient condition for the existence of the Fourier transform of a sequence $x[n]$ is that it be absolutely summable: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. In other words, the Fourier transform exists if the sum $\sum_{n=-\infty}^{\infty} |x[n]|$ converges. The Fourier transform may however exist for sequences where this is not true — a rigorous mathematical treatment can be found in the theory of **generalised functions**.

8 Symmetry properties of the Fourier transform

Any sequence $x[n]$ can be expressed as

$$x[n] = x_e[n] + x_o[n],$$

where $x_e[n]$ is **conjugate symmetric** ($x_e[n] = x_e^*[-n]$) and $x_o[n]$ is **conjugate antisymmetric** ($x_o[n] = -x_o^*[-n]$). These two components of the sequence can be obtained as:

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$$
$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n].$$

If a real sequence is conjugate symmetric, then it is an **even** sequence, and if conjugate antisymmetric, then it is **odd**.

Similarly, the Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate symmetric and antisymmetric parts:

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}),$$

where

$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})]$$
$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})].$$

With these definitions, and letting

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}),$$

the symmetry properties of the Fourier transform can be summarised as follows:

Sequence $x[n]$	Transform $X(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_e(e^{j\omega})$
$j\,\mathrm{Im}\{x[n]\}$	$X_o(e^{j\omega})$
$x_e[n]$	$X_R(e^{j\omega})$
$x_o[n]$	$jX_I(e^{j\omega})$

Most of these properties can be proved by substituting into the expression for the Fourier transform. Additionally, for real $x[n]$ the following also hold:

Real sequence $x[n]$	Transform $X(e^{j\omega})$
$x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
$x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$
$x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$
$x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
$x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$
$x_e[n]$	$X_R(e^{j\omega})$
$x_o[n]$	$jX_I(e^{j\omega})$

9 Fourier transform theorems

Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. The following theorems then apply:

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Some useful Fourier transform pairs are:

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n + 1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$