EEE4114F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2019
3 hours

Information

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *six* questions totalling 60 marks. You must answer all of them.
- **Part B** has *five* questions totalling 40 marks. You must answer all of them.
- Part A and part B must be answered in separate exam books.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- You have 3 hours.

PART A

Digital signal processing

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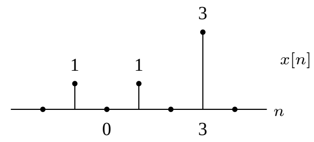
1. Suppose $x[n]$ is the signal below:



Plot the following:

(a) $x_1[n] = x[2n-1]$

(b) $x_2[n] = x[-2(n-1)]$

(c) $x_3[n] = -x[-n-1]$

(d) $x_4[n] = (\delta[n] - \delta[n-1]) * x[n]$

(e) $x_5[n] = x[n] * x[-n]$

(f) $x_6[n] = \sum_{k=-\infty}^{n-1} x[k]$.

(15 marks)

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2. The input $x[n]$ and output $y[n]$ of a system obey the relationship

$$y[n] = \sum_{k=-2}^{1} x[n-k-1].$$

(a) Show that the system is linear.

(b) The system is time invariant. Find and sketch the impulse response.

(c) Is the system causal? Why?

(d) Is the system stable? Why?

(e) Find the output of the system when the input is the unit step $u[n]$.

(10 marks)

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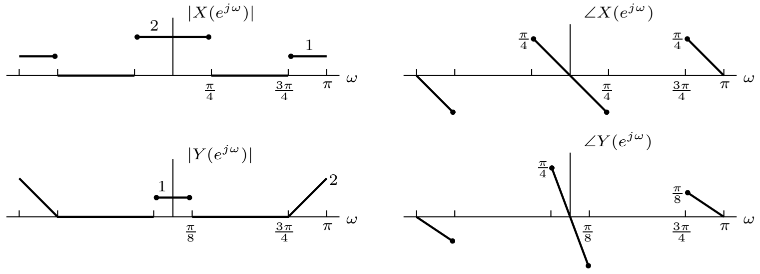
3. (a) A real signal $x[n]$ is passed through an LTI system with impulse response $h[n]$ and frequency response $H(e^{j\omega})$ to produce a real output $y[n]$. Below are sketches of the magnitude and phase of $X(e^{j\omega})$ and $Y(e^{j\omega})$:



Sketch the magnitude and phase of $H(e^{j\omega})$. Clearly label any regions which cannot be determined from the given information. Label the numerical values of the endpoints of each segment of the graph, as in the above examples.

(b) Assume an LTI system with impulse response $h[n]$. Suppose that we observe the input $x_i[n]$ and the corresponding output $y_i[n]$, such that $y_i[n] = h[n] * x_i[n]$. For which input signals below is it possible to determine $h[n]$ from this information? In all cases given an explanation of your answer.

i. $x_1[n] = \delta[n-3]$

ii. $x_2[n] = \frac{\sin(\pi n/10)}{\pi n/10}$

iii. $x_3[n] = e^{j\pi n/17}$

iv. $x_4[n] = \left(\frac{1}{2}\right)^n u[n]$.

(10 marks)

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4. Consider a causal linear time-invariant system which results in the output

$$y[n] = \left(\frac{1}{3}\right)^n u[n]$$

when the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

Determine a closed-form expression for the impulse response $h[n]$ of the system. Use your result to sketch the first three samples of the impulse response $h[0]$, $h[1]$, and $h[2]$.

(10 marks)

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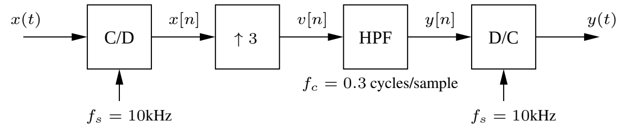
5. Consider the following block diagram of a system:



An input signal $x(t)$ is discretised by an ideal continuous-to-discrete converter at the sampling frequency $f_s = 10$kHz. It is then upsampled by a factor 3, passed through an ideal highpass filter with cutoff frequency $f_c = 0.3$ cycles/sample, and reconstructed by an ideal discrete-to-continuous converter operating at 10kHz. The upsampling block performs no filtering. Suppose the input is $x(t) = \cos(5000t)$. Determine and sketch the spectrum of each signal in the diagram.

Note the continuous-time Fourier pair $e^{j\Omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\Omega - \Omega_0)$.

(10 marks)

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6. Two real-valued signals $x_1[n]$ and $x_2[n]$ for $n = 0, 1, 2, 3$ are combined as $x[n] = x_1[n] + jx_2[n]$. You measure $X[k]$, the DFT of $x[n]$, as

$$X[0] = 2 + j2, \quad X[1] = -j4, \quad X[2] = 2 - j2, \quad X[3] = 0.$$

What is $x_2[n]$ for $n = 0, 1, 2, 3$?

(5 marks)

PART B

Machine learning

1. We have collected n data points (all positive) from a data-source which is known to follow a Gaussian distribution. These collected data are known to be IID (independent and identically distributed).

   For a Gaussian distribution, the pdf for observation x is given by

   $$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right).$$

   Find the maximum likelihood estimation (MLE) for the parameter μ.

   Suggested procedure: First find the likelihood of the parameter μ given the n observations; then find the log-likelihood; next find the value of the parameter which will maximise the log-likelihood.

   (9 marks)

2. Imagine that machines have gone really intelligent and somehow destroyed all humans. Then a machine discovers two thermometers, one measuring temperature in Celsius and one in Fahrenheit. The machine is confused as it does not know the relation that links these two measurements, so it gathers a set of 7 measurements:

   $$-40, -10, 0, 8, 15, 22, 38$$

   in Celsius corresponding to

   $$-40, 14, 32, 46, 59, 72, 100$$

   in Fahrenheit. It builds an ANN model to convert Celsius value to Fahrenheit using the following:

```
l0 = tf.keras.layers.Dense(units=50, activation='relu', input_shape=[1])
l1 = tf.keras.layers.Dense(units=1)
model = tf.keras.Sequential([l0, l1])
```

   When tested with new measurements this model does not perform well, giving an error of around 2–3 degrees! Can you explain why?

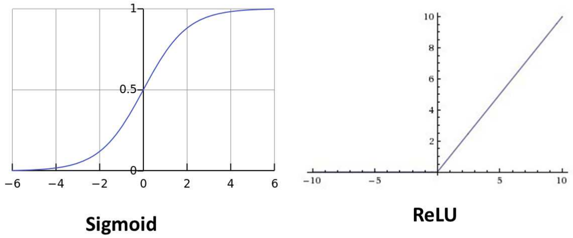
   Suggested procedure: first draw a rough plot of the data available; comment on the complexity of the task; now look at the code and explain what each layer means; then comment on what might be going wrong.

   (9 marks)

3. In a game two dice are rolled and the value of the higher number is noted down. If both dice show the same number then that number is also noted down. What is the expected value of the noted-down figure?

   (9 marks)

4. The figure below shows two common activation functions, namely sigmoid and ReLU.



   (a) What is an activation function?

   (b) Which of these two functions is differentiable at all points?

   (c) Which of them is more popular for deep learning and why? Give at least two reasons.

(9 marks)

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5. What is a convolutional neural network? Either use mathematical notation or diagram to show the process.

(4 marks)

Transforms

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} X(j\Omega)e^{j\Omega t}d\Omega \qquad \text{(CTFT)}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} \qquad \text{(DTFT)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \qquad \text{(ZT)}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \quad \text{with} \quad W_N = e^{-j\frac{2\pi}{N}} \qquad \text{(DFT)}$$

Discrete-time Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common discrete-time Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Z-transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(z), Y(z)$	ROC	Property
$ax[n] + by[n]$	$aX(z) + bY(z)$	ROC contains $R_x \cap R_y$	Linearity
$x[n - n_d]$	$z^{-n_d} X(z)$	ROC $= R_x$	Time shift
$z_0^n x[n]$	$X(z/z_0)$	ROC $= \|z_0\| R_x$	Frequency scale
$x^*[-n]$	$X^*(1/z^*)$	ROC $= \frac{1}{R_x}$	Time reversal
$nx[n]$	$-z\frac{dX(z)}{dz}$	ROC $= R_x$	Frequency diff.
$x[n] * y[n]$	$X(z)Y(z)$	ROC contains $R_x \cap R_y$	Convolution
$x^*[n]$	$X^*(z^*)$	ROC $= R_x$	Conjugation

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n) u[n]$	$\frac{1-\cos(\omega_0) z^{-1}}{1-2\cos(\omega_0) z^{-1} + z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n) u[n]$	$\frac{1-r\cos(\omega_0) z^{-1}}{1-2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}$	$\|z\| > \|r\|$

Other

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \quad \text{for} \quad a \neq 1.$$