EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2016
3 hours

Information

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *six* questions totalling 50 marks. You must answer all of them.
- **Part B** has *seven* questions totalling 50 marks. You must answer all of them.
- Parts A and B must be answered in different sets of exam books, which will be collected separately.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- You have 3 hours.

PART A

Digital signal processing

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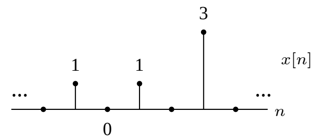
1. If $x[n]$ is the signal



then plot the following:

(a) $y_1[n] = x[2-n]$

(b) $y_2[n] = \sum_{k=-\infty}^{n} x[k]$

(c) $y_3[n] = u[n] * x[n]$

(d) $y_4[n] = x[n] - x[n-1]$

(e) $y_5[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-1]$.

(10 marks)

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2. The signal $x[n] = u[n+1] - u[n-1]$ is input to a causal LTI system described by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

(a) Find the output signal $y[n]$.

(b) Draw a pole-zero plot of the system.

(c) Is the system stable? Why?

(10 marks)

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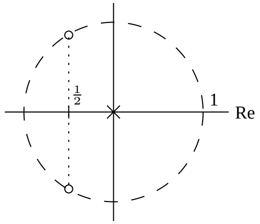
3. A system has the following pole-zero plot:



It is known that when the input is $x[n] = 1$ for all n then the output is $y[n] = 1$ for all n.

(a) Sketch the impulse response $h[n]$ of the system.

(b) Sketch the magnitude of the frequency response for the system.

(c) Find the approximate phase of the frequency response for $\omega = \pi/2$ rad/sample.

(10 marks)

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4. For each system described below, identify the transfer function of the inverse system, and determine whether it can be both causal and stable:

(a) $H(z) = \frac{1-8z^{-1}+16z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$,

(b) $H(z) = \frac{z^2-\frac{81}{100}}{z^2-1}$,

(c) $h[n] = 10(\frac{-1}{2})^n u[n] - 9(\frac{-1}{4})^n u[n]$.

(10 marks)

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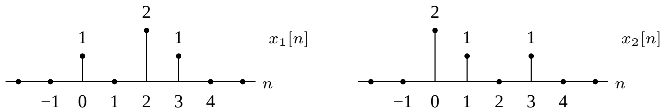
5. (a) Find the 4-point circular convolution of the signals below:



(b) The DFTs of two 4-point sequences $y_1[n]$ and $y_2[n]$ are

$$Y_1[k] = \{2, 2+j, -2, 2-j\} \quad \text{and} \quad Y_2[k] = \{2, 2, 6, 2\}.$$

Find the (4-point) circular convolution of y_1 and y_2 (that is, find the time domain values of the circular convolution).

(c) Explain how linear convolution can be done using circular convolution.

(5 marks)

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6. Consider a system where the product $x(t)$ of two continuous-time signals $x_1(t)$ and $x_2(t)$ (that is, $x(t) = x_1(t)x_2(t)$) is sampled by a periodic impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

Denote the sampled signal by $x_p(t)$, and suppose the two input signals are band limited:

$$X_1(j\Omega) = 0, \quad |\Omega| \geq \Omega_1, \qquad \text{and} \qquad X_2(j\Omega) = 0, \quad |\Omega| \geq \Omega_2.$$

(a) Derive a mathematical expression of this impulse-train sampling by showing how $X_p(j\Omega)$ is related to $X_1(j\Omega)$ and $X_2(j\Omega)$.

(b) Determine the maximum sampling interval T_M such that $x(t)$ can be reconstructed from $x_p(t)$ by using an ideal lowpass filter.

(c) Specify the impulse response of the ideal low pass filter in part (b).

The following are valid continuous-time Fourier pairs, where $u(\cdot)$ denotes the unit step:

$$\tau \text{sinc}\frac{\tau t}{2\pi} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi[u(\Omega + \tau/2) - u(\Omega - \tau/2)]$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{2\pi}{T}).$$

(5 marks)

PART B

Wavelets and frames

Problem 1:

Let the support of $\varphi(t)$ be the interval $[-1, 1]$. Let

$$\varphi(t) = \begin{cases} t+1 & -1 < t < 0 \\ -t+1 & 0 < t < 1. \end{cases}$$

a. Determine the L_2-norm $||\varphi(t)||_2$. (2 marks)

Denote $\frac{\varphi(t)}{||\varphi(t)||_2}$ by $\tilde{\varphi}(t)$,

$$\tilde{\varphi}(t) = \frac{1}{||\varphi(t)||_2}\varphi(t).$$

b. Write down the $2-$scale dilation equation for $\tilde{\varphi}(t)$. (2 marks)

c. Determine the h_n parameters in the dilation equation. (2 marks)

d. Determine $H(\omega)$, the Fourier transform of the discrete set of coefficients h_n. (2 marks)

e. Plot $|H(\omega)|^2$ as a function of ω. (2 marks)

(Total: 10 marks)

Problem 2:

(i) Let $\int\limits_{-\infty}^{\infty} dt\varphi(t)$ be finite. Consider the dilation equation

$$\varphi(t) = \sum_{n=-\infty}^{\infty} h_n \sqrt{2}\varphi(2t-n).$$

(ii) Integrate both sides of this equation from $-\infty$ to ∞.

What condition can be concluded for the coefficients h_n when considering (i) and (ii)?

(Total: 6 marks)

Problem 3:

(i) Consider the dilation equation

$$\varphi(t) = \sum_{n=-\infty}^{\infty} h_n \sqrt{2}\varphi(2t-n).$$

(ii) Assume $\varphi(t)$ is orthonormal to its integer translates:

$$\int\limits_{-\infty}^{\infty} dt\varphi(t)\varphi(t-k) = \delta_{0k}.$$

What conditions can be concluded for the coefficients h_n when considering (i) and (ii)?

(Total: 6 marks)

Problem 4:

Consider

$$\varphi(t) = \sum_{n=0}^{5} h_n \sqrt{2}\varphi(2t-n).$$

a. Determine the support of the function $\varphi(t)$. (2 marks)

b. Determine the value of the function $\varphi(t)$ at $t = 0$; i.e., $\varphi(0)$. (2 marks)

c. Determine the value of the function $\varphi(t)$ at $t = 5$; i.e., $\varphi(5)$. (2 marks)

(Total: 6 marks)

Problem 5:

Denote the Fourier transform of a fairly general function $f(t)$ by $F(\omega)$. Construct

$$G(\omega) = \frac{F(\omega)}{\sqrt{\sum\limits_{n=-\infty}^{\infty} |F(\omega + 2\pi k)|^2}}.$$

Denote the inverse Fourier transform of $G(\omega)$ by $g(t)$.

a. Determine the value of the integral:

$$\int\limits_{-\infty}^{\infty} dt g^*(t-5) g(t-7) = \, ?$$

(3 marks)

b. Determine the value of the integral:

$$\int\limits_{-\infty}^{\infty} dt g^*(t-15) g(t-15) = \, ?$$

(3 marks)

In (a) and (b) above the asterisk denotes complex conjugate.

(Total: 6 marks)

Problem 6:

Consider the vectors

$$|f_1 >= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |f_2 >= \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad |f_3 >= \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

a. Construct the dual frame vectors corresponding to the frame vectors given above.

(3 marks)

b. Express the resolution of identity in terms of the frame and the constructed dual frame vectors. (3 marks)

(Total: 6 marks)

Problem 7:

Consider $\mathcal{N}$ vectors $|f_1 >, |f_2 >, \cdots, |f_{\mathcal{N}} >$ in an $N-$dimensional vector space, with $\mathcal{N} > N$. Consider the relationships:

$$A||f(t)||^2 \leq \sum_{n=1}^{\mathcal{N}} | < f_n(t)|f(t) > |^2 \leq B||f(t)||^2$$

with $0 < A \leq B < \infty$.

Express the above in a form which only involves the frame operator and is thus independent of $f(t)$.

(Total: 10 marks)

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$