EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2013
3 hours

Information

- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *six* questions totalling 50 marks. You must answer all of them.
- **Part B** has *ten* questions totalling 50 marks. You must answer all of them.
- Parts A and B must be answered in different sets of exam books, which will be collected separately.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- You have 3 hours.

PART A

Basic digital signal processing theory.

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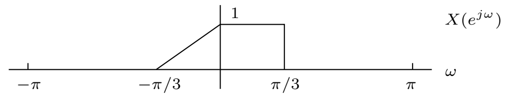
1. A sequence $x[n]$ has a zero-phase DTFT $X(e^{j\omega})$ given below:



Sketch the DTFT of the sequence $2x[n]e^{-j\pi n/3}$.

(5 marks)

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2. Find the impulse response corresponding to the system function

$$H(z) = \frac{7z^2 - 4z}{z^2 - \frac{3}{2}z - 1}$$

for each possible region of convergence. In each case comment on the causality and stability properties of the system.

(10 marks)

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3. Let $x[n]$ be a discrete-time signal obtained by sampling the continuous signal $x(t)$ at a sampling rate $f_s = 1/T$ Hz:

$$x[n] = x(nT).$$

Assume that no aliasing occurs. Describe by sketching a block diagram and providing a clear explanation, how you would implement a *discrete-time* system with input $x[n]$ and output $y[n]$ that delays $x[n]$ by half a sample, so

$$y[n] = x(nT - T/2).$$

Hint: make use of upsamplers, downsamplers, and ideal filters.

(5 marks)

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4. Consider the following discrete-time sequences:

$$x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2] + 5\delta[n-3]$$
$$y[n] = \delta[n] - 2\delta[n-3].$$

(a) Write an expression for the 4-point DFT $X[k]$ of $x[n]$, and find the value $X[1]$.

(b) Find the 4-point inverse DFT of $X[k]W_4^k$, where $X[k]$ is the 4-point DFT of $x[n]$ and $W_4 = e^{-j\frac{2\pi}{4}}$.

(c) Find the 4-point circular convolution of $x[n]$ with $y[n]$.

(d) Explain how you would calculate the result for part (c) using the fast Fourier transform (FFT).

(e) How could you use the FFT to calculate the linear convolution of $x[n]$ and $y[n]$?

(10 marks)

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5. A discrete-time, causal, linear time-invariant filter $H(z)$ has

six zeros located at: $z = e^{\pm j\pi/8}, z = e^{\pm j7\pi/8}, z = \pm 1$

and six poles located at: $z = \pm j0.95, z = 0.95e^{\pm j9\pi/20}, z = 0.95e^{\pm j11\pi/20}$.

(a) Plot the pole-zero diagram of $H(z)$ in the z-plane and provide its region of convergence.

(b) Sketch the magnitude response $|H(e^{j\omega})|$ directly from the pole-zero plot, and indicate the approximate gain at $\omega = \pi/2$.

(c) What type of frequency-selective filter is $H(e^{j\omega})$? Explain your answer.

(d) Answer the following questions, explaining your answers:

i. Is $H(z)$ an IIR or FIR filter?

ii. Is $H(z)$ a stable filter?

iii. Is $h[n]$, the impulse response of the filter, a real function?

(10 marks)

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6. Consider a causal LTI system with the system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}}$$

where a is real.

(a) Write a difference equation that relates the input and the output of this system

(b) For what range of values of a is the system stable?

(c) For $a = 1/2$ plot the pole-zero diagram and show the ROC.

(d) Find the impulse response $h[n]$ for this system for $a = 1/2$.

(e) Determine and plot the magnitude response of this system for $a = 1/2$. What type of system is it?

(10 marks)

PART B

Wavelets and frames.

P1: Let the function $f(t)$ be defined by

$$f(t) = \begin{cases} \sin(n\pi t) & -1 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

P1-a: Calculate $\|f(t)\|_2$, the L_2-norm of $f(t)$.

(2 marks)

P1-b: Let $\widetilde{f}(t)$ denote $f(t)$ normalized; i.e., $\|\widetilde{f}(t)\|_2 = 1$. Write down the expression for $\widetilde{f}(t)$.

(2 marks)

P2: Consider the following complete set of orthonormal functions on the interval $(-1, 1)$:

$$\frac{1}{\sqrt{2}}, \{\cos(n\pi t)\,|n \in \mathbb{N}\}, \{\sin(n\pi t)\,|n \in \mathbb{N}\} \tag{2}$$

Utilizing Dirac's bracket notation, consider the following resolution of identity for the L_2-space of functions with support $(-1, 1)$:

$$\mathbb{I} = |\frac{1}{\sqrt{2}} >< \frac{1}{\sqrt{2}}| \quad + \quad \sum_{n\in\mathbb{N}} |\cos(n\pi t) >< \cos(n\pi t)| \\ + \quad \sum_{n\in\mathbb{N}} |\sin(n\pi t) >< \sin(n\pi t)| \tag{3}$$

Let the function $f(t)$ satisfy Dirichlet's conditions on the interval $(-1, 1)$, and be zero outside this interval. In bracket notation write $|f(t)>$ for $f(t)$. Operate (3) from the left onto the function $|f(t)>$ to obtain:

$$\mathbb{I}|f(t) >= |\frac{1}{\sqrt{2}} >< \frac{1}{\sqrt{2}}|f(x)> \quad + \quad \sum_{n\in\mathbb{N}} |\cos(n\pi t) >< \cos(n\pi t)\,|f(t)> \\ + \quad \sum_{n\in\mathbb{N}} |\sin(n\pi t) >< \sin(n\pi t)\,|f(t)> \tag{4}$$

It is self-evident that certain groups of terms in (4) vanish for even- or odd-functions, and thus the Eq. (4) simplifies for such functions.

P2-a: Simplify the expression on the right-hand side of the Eq. (4) for functions $f(t)$ satisfying the condition $f(-t) = f(t)$ on the interval $(-1, 1)$.

(2 marks)

P2-b: Deduce from your result obtained in **P2-a** the expression for the resolution of identity, which characterizes the space of even functions with support $(-1, 1)$.

(2 marks)

P2-c: Simplify the expression on the right-hand side of the Eq. (4) for functions $f(t)$ satisfying the condition $f(-t) = -f(t)$ on the interval $(-1, 1)$.

(2 marks)

P2-d: Deduce from your result obtained in **P2-c** the expression for the resolution of identity, which characterizes the space of odd functions with support $(-1, 1)$.

(2 marks)

P3: Let the functions $\varphi(t)$ and $\psi(t)$ denote the scaling function and the wavelet for a Multiresolution Analysis (MRA) in Hilbert space. Let the function $\varphi(t)$ generate the function space ν_0. Let the function $\psi(t)$ and its compressed versions generate the spaces $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \cdots$.

Assume the following representation for $f(t)$ is valid:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi(t-k) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d_{j,k} 2^{\frac{j}{2}} \psi(2^j t - k) \qquad (5)$$

Consider the right-hand side of (5).

P3-a: Why is the first term a single series?

P3-b: Why is the second term (following the summation sign) a double series?

P3-c: Write down the expression for c_k.

P3-d: Write down the expression for $d_{j,k}$.

P3-e: Utilize Dirac's bracket notation, and consider the results obtained in **P3-c** and **P3-d**. In the light of your results deduce the expression for the resolution of identity from (5).

P3-f: Let j run from $-\infty$ to ∞. Considering the result obtained in the previous step, deduce the expression for the resolution of identity, when j varies from $-\infty$ to ∞.

(6 marks)

P4-a: Given the general ''low pass'' filter coefficient $h(n)$ write down the two-scale dilation equation for the scaling function.

(2 marks)

P4-b: Given the general ''high pass'' filter coefficients $g(n)$ write down the two-scale dilation equation for the wavelet.

(2 marks)

P5-a: Determine the filter coefficients $h(n)$ for the triangle (piece-wise linear) scaling function.

(2 marks)

P5-b: The coefficients $h(n)$, characterizing the triangle (piece-wise linear) scaling function, constitute a ''low pass'' filter. Why?

(2 marks)

P5-c: Determine the filter coefficients $g(n)$ for the triangle (piece-wise linear) wavelet.

(2 marks)

P5-d: The coefficients $g(n)$, characterizing the triangle (piece-wise linear) wavelet, constitute a ''high pass'' filter. Why?

(2 marks)

P6: Given a fairly general function $f(t)$. Apply Meyer's orthogonalization technique to $f(t)$.

(4 marks)

P7: Using the general dilation equation for the wavelet $\psi(t)$, express the three-generations compressed normalized wavelet

$$2^{\frac{3}{2}} \psi(2^3 t - m)$$

in terms of $2^{\frac{4}{2}} \varphi(2^4 t - n)$ summed over n.

(4 marks)

P8: The Mexican-hat wavelet $\mathcal{M}_h(t)$ can be obtained by taking the second derivative of the negative Gaussian function:

$$\mathcal{G}(t) = -\frac{1}{2} e^{-t^2}$$

Sketch the Mexican-hat wavelet $\mathcal{M}_h(t)$.

(2 marks)

P9: Ordinarily signal-analysis and signal -synthesis are carried out by using a system of orthonormal basis (ONB) functions. However, if the orthonormality condition of the analysis basis functions is violated, a system of dual basis functions is required for accomplishing the synthesis of signals. The following problem illustrates the content of this concept in terms of vectors.

Let $|\mathbf{e}_1>$ and $|\mathbf{e}_2>$ be unit normal vectors in the $(x, y)-$plane.

Let the vectors $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$ be defined by the following equations:

$$|\mathbf{f}_1> \;=\; 2|\mathbf{e}_1> +1|\mathbf{e}_2> \qquad (6a)$$

$$|\mathbf{f}_2> \;=\; 2|\mathbf{e}_1> +4|\mathbf{e}_2> \qquad (6b)$$

Evidently, the vectors $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$ are neither normal nor orthogonal.

Provide a sketch of the vectors $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$.

Construct the dual vectors $<\widetilde{\mathbf{f}}_1|$ and $<\widetilde{\mathbf{f}}_2|$ corresponding to $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$, respectively, first graphically and then analytically.

Employ Dirac's bracket notation.

Resolve the identity operator $\mathbb{I}$ in the plane (i.e., the 2×2 unity matrix) in terms of the ket-vectors $|\mathbf{f}_1>$ and $|\mathbf{f}_2>$ and their dual bra-vectors $<\widetilde{\mathbf{f}}_1|$ and $<\widetilde{\mathbf{f}}_2|$.

(3 marks)

P10: In the foregoing problem it was mentioned that customarily signal-analysis and signal-synthesis are carried out by using a system of orthonormal basis (ONB) functions. However, if the analysis functions are over-complete (they constitute a frame), a system of over-complete functions (dual frames) is required for accomplishing the synthesis of signals. The following problem illustrates the content of this concept in terms of vectors.

Let $|\mathbf{e}_1>$ and $|\mathbf{e}_2>$ denote unit normal vectors in the $(x, y)-$plane.

Let the ket vectors $|\mathbf{f}_1>$, $|\mathbf{f}_2>$ and $|\mathbf{f}_3>$ be defined by the following equations:

$$|\mathbf{f}_1> \;=\; |\mathbf{e}_1> \qquad (7a)$$

$$|\mathbf{f}_2> \;=\; |\mathbf{e}_1> -|\mathbf{e}_2> \qquad (7b)$$

$$|\mathbf{f}_3> \;=\; |\mathbf{e}_1> +|\mathbf{e}_2> \qquad (7c)$$

The over-complete set of vectors $|\mathbf{f}_1>$, $|\mathbf{f}_2>$ and $|\mathbf{f}_3>$ constitutes a frame.

The dual frame (bra vectors) $<\widetilde{\mathbf{f}}_1|$, $<\widetilde{\mathbf{f}}_2|$ and $<\widetilde{\mathbf{f}}_3|$ are given as follows:

$$<\widetilde{\mathbf{f}}_1| \;=\; \frac{1}{3} < \mathbf{e}_1| \qquad (8a)$$

$$<\widetilde{\mathbf{f}}_2| \;=\; \frac{1}{3} < \mathbf{e}_1| - \frac{1}{2} < \mathbf{e}_2| \qquad (8b)$$

$$<\widetilde{\mathbf{f}}_3| \;=\; \frac{1}{3} < \mathbf{e}_1| + \frac{1}{2} < \mathbf{e}_2| \qquad (8c)$$

Resolve the identity operator $\mathbb{I}$ in the plane (i.e., the 2×2 unity matrix) in terms of the frame vectors $|\mathbf{f}_1>$, $|\mathbf{f}_2>$ and $|\mathbf{f}_3>$ and their dual frame vectors $<\widetilde{\mathbf{f}}_1|$, $<\widetilde{\mathbf{f}}_2|$ and $<\widetilde{\mathbf{f}}_3|$.

(7 marks)

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\|z\| < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$\|z\| > r$