EEE4001F EXAM
DIGITAL SIGNAL PROCESSING

University of Cape Town
Department of Electrical Engineering

June 2011
3 hours

Information

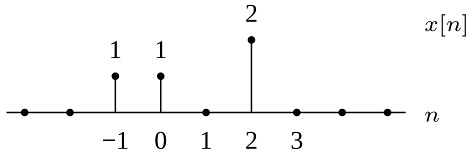
- The exam is closed-book.
- There are two parts to this exam.
- **Part A** has *seven* questions totalling 70 marks. You must answer all of them.
- **Part B** has *two* questions, each counting 15 marks. You must answer *both* of them.
- A table of standard Fourier transform and z-transform pairs appears at the end of this paper.
- A formula sheet for the radar/sonar question appears at the end of this paper.
- You have 3 hours.

PART A

Answer all of the following questions.

---

1. If $x[n]$ is the signal below



then plot the following:

(a) $y_1[n] = x[1-n]$

(b) $y_2[n] = x[-2n+1]$

(c) $y_3[n] = x[n] - x[n-1]$

(d) $y_4[n] = \sum_{k=-\infty}^{n} x[k]$

(e) $y_5[n] = x[n] * u[n]$.

(10 marks)

---

2. Consider the following linear constant coefficient difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1].$$

Determine $y[n]$ when $x[n] = \delta[n]$ and $y[n] = 0$ for $n < 0$.

(10 marks)

---

3. Consider the linear time-invariant system described by the following transfer function:

$$H(z) = \frac{z-1}{z}.$$

(a) This system is known as a *backwards Euler differentiator*. Why do you think it has been given this name? Motivate and elaborate.

(b) Determine an expression for the magnitude of the system's frequency response $H(e^{j\omega})$. Plot the magnitude over the interval $0 \leq \omega \leq 2\pi$. What kind of filter does this system represent?

(c) Sketch the phase of the system's frequency response $H(e^{j\omega})$ over the interval $0 \leq \omega \leq 2\pi$.

(10 marks)

---

4. Consider two discrete-time signals $x_1[n] = \delta[n] + 2\delta[n-2] + \delta[n-3]$ and $x_2[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-3]$.

(a) Determine and plot the linear convolution of $x_1[n]$ with $x_2[n]$.

(b) Determine and plot the 4-point circular convolution of $x_1[n]$ with $x_2[n]$.

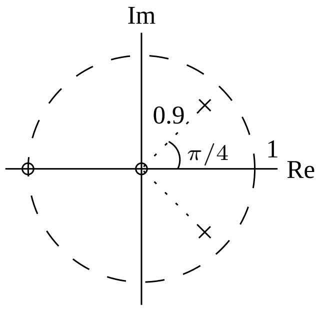
(c) How would you calculate the linear convolution result using a circular convolution operation?

(d) Why would you want to implement linear convolution using a circular convolution operation?

(10 marks)

---

5. The following figure shows the pole-zero plot of a system with two poles and two zeros:



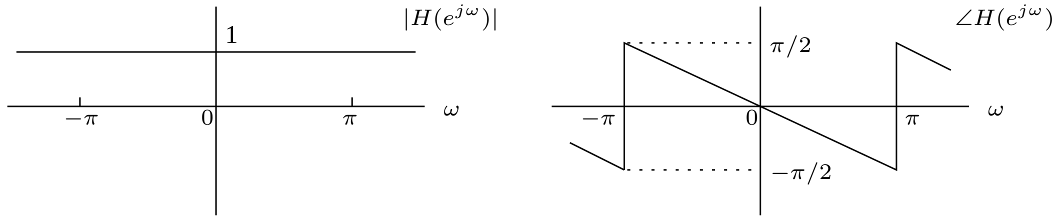
(a) Determine the transfer function $H(z)$ describing this system assuming that it has a DC gain of one.

(b) Sketch the magnitude of the frequency response of the system.

(10 marks)

---

6. A discrete-time LTI system has the following magnitude and phase response:



(a) Determine the output if the input is the signal $x[n] = e^{j\frac{5\pi}{2}n}$.

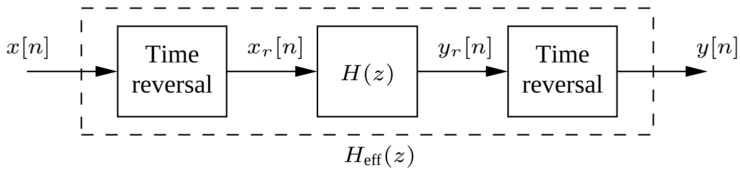
(b) Determine and sketch the output if the input is the signal $x[n] = \cos\left(\frac{5\pi}{2}n\right)$.

(10 marks)

---

7. (a) Suppose $x_r[n]$ is a time reversal of the signal $x[n]$, so $x_r[n] = x[-n]$. Show that $X_r(z) = X(1/z)$ in the z-transform domain.

(b) Now consider the system below:



Show that the effective transfer function linking the input $X(z)$ to the output $Y(z)$ is $H_{\text{eff}}(z) = H(1/z)$.

(c) If $h[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} H(z)$ and $Y(z) = H(1/z)X(z)$, find a time-domain expression for $y[n]$ in terms of $x[n]$ and $h[n]$.

(10 marks)

PART B

Answer *both* of the following two questions. Each question counts 15 marks.

---

1. **Image processing and computer vision**

   (a) Assuming that

   $$x(n_1, n_2) = \delta(n_1, n_2) + \delta(n_1 - 1, n_2) + \delta(n_1, n_2 - 1),$$

   find and plot $y(n_1, n_2) = x(n_1, n_2) * x(n_1, n_2)$.

   (b) The two-dimensional convolution of the signal $x(n_1, n_2)$ with the kernel $h(n_1, n_2)$ is given by

   $$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h(k_1, k_2) x(n_1 - k_1, n_2 - k_2).$$

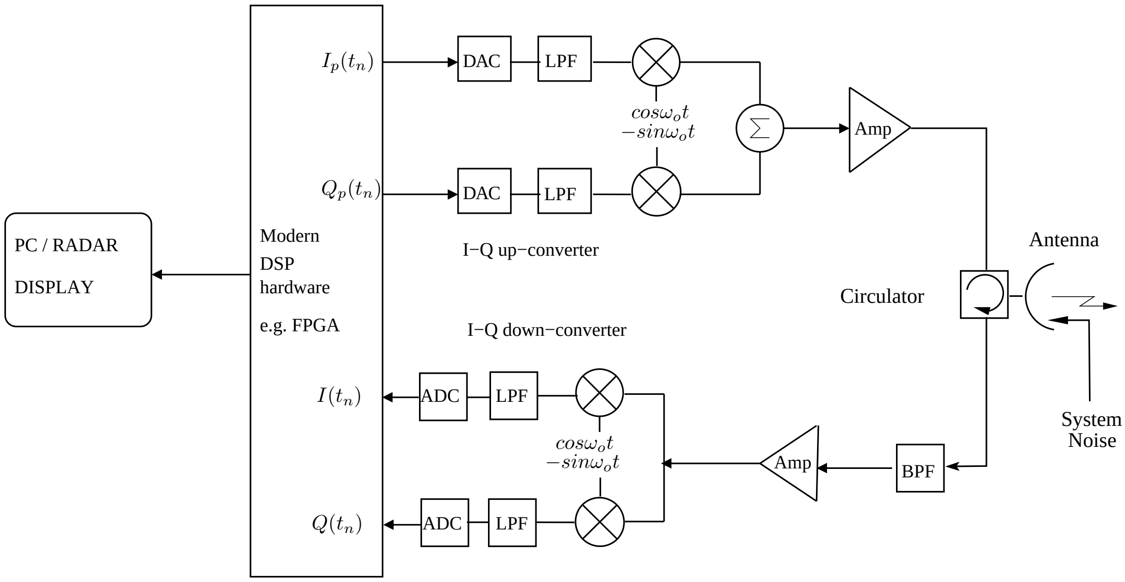
   If the kernel is separable then we can write $h(k_1, k_2) = h_1(k_1)h_2(k_2)$. Show that in this case the 2-D convolution can be implemented as a set of 1-D convolution operations, and indicate how the computation required to implement 2-D convolution can be reduced if the convolution kernel satisfies this separability property.

   (c) Explain how the second derivative of an image can be used to formulate an edge detector. How would you estimate the required first and second derivatives in a discrete setting? How would you reduce the effect of noise in the process?

---

2. **Radar/sonar signal processing**

A simplified block diagram of a radar is shown below:



(a) Draw a neatly labelled block diagram of an equivalent *analytic* (complex) signal model of the radar.

(2 marks)

(b) Illustrate with the aid of sketches of the *frequency spectra*, how the signals in the system are related, particularly illustrate the relationship between the *frequency* spectra of the following signals:

i. the impulse response of the scene $\xi(t) \leftrightarrow \xi_f(f)$

ii. the transmitted rf pulse $v_{tx}(t) \leftrightarrow V_{tx}(f)$ and baseband form $p(t) \leftrightarrow P(f)$

iii. complex baseband signal $v_{bb}(t) \leftrightarrow V_{bb}(f)$.

(3 marks)

(c) A digital signal processing algorithm must be developed for pulse compression and display of the echoes received by the radar. The transmitted pulse is a *chirp* pulse with bandwidth of 80 MHz, and a centre frequency of 8 GHz. The processor operates on the baseband signals sampled from the IQ down-converter.

i. What sample rate is required for the signals at the output of the IQ down-converter?

ii. What digital signal processing steps would you carry out to obtain a processed "range profile" of the scene, considering that you would like to optimize *the signal to noise ratio*?

iii. Calculate the 3dB range resolution in metres.

(5 marks)

(d) If a *deconvolution (inverse)* filter is used to process the radar data in (c), sketch the point target response (both magnitude and phase) as a function of range, that you would expect to see at the output for a point scatterer at a range of 800m.

(3 marks)

(e) What additional processing steps can one implement to improve the sidelobe levels of the point target response? Sketch a typical output and indicate clearly the effect of such processing, compared to the output of the deconvolution filter in (d).

(2 marks)

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

FORMULA SHEET V5 EEE4001F 2011

PLEASE REPORT ANY ERRORS TO A.J.W.

Fourier Relationships

$x(t) \leftrightarrow X(f)$

$x(t-t_o) \leftrightarrow X(f)e^{-j2\pi f t_o}$

$x(t)e^{-j2\pi f_o t} \leftrightarrow X(f+f_o)$

$x^*(t) \leftrightarrow X^*(-f)$

$BSa(\pi\beta t) \leftrightarrow rect(\frac{f}{B})$

$rect\left(\frac{t}{\tau}\right) \leftrightarrow \tau Sa(\pi f \tau)$

$\delta(t) \leftrightarrow 1$

For any 'real' signal $x(t)$, $X(-f) = X^*(f)$

Convolution $x(t) \otimes h(t) \leftrightarrow X(f)H(f)$

Radar Equation

$P_r = \frac{P_t G_t \sigma A_e}{(4\pi R^2)^2}$ where $A_e = \frac{G_r \lambda^2}{4\pi}$

IQ Down-converter

$I(t) = [2x(t)\cos(\omega_o t)]_{LPF}$

$Q(t) = [-2x(t)\sin(\omega_o t)]_{LPF}$

$V(t) = I(t) + jQ(t) \leftrightarrow V(f) = 2X^+(f+f_o)$

Matched Filter General

$H(f) = \frac{X^*(f)}{S_{n_i}(f)} \rightarrow X^*(f)$ (white noise)

$\frac{|\upsilon_o(t_{peak})|^2}{\left|\overline{n_o(t)}\right|^2} = \frac{E}{\eta/2}$ (white noise)

ANALYTIC RADAR MODEL

Baseband Pulse $p(t)$

Transmitted $\upsilon_{TX}(t) = p(t)e^{j2\pi f_o t}$

EXTENDED TARGET RESPONSE

$\upsilon_{RX}(t) = \int_{\tau=-\infty}^{\infty} \zeta(\tau)\upsilon_{TX}(t-\tau)d\tau = \zeta(t) \otimes \upsilon_{TX}(t)$

$|\zeta(\tau)|^2 \propto \frac{1}{R^4(\tau)} |\beta(\tau)|^2$

$V_{RX}(f) = \zeta(f)V_{TX}(f)$

Baseband Signal

$\upsilon_{bb}(t) = \left[\upsilon_{RX}(t)e^{-j2\pi f_o t}\right] \otimes h_{bb}(t) + n_{bb}(t)$

$\upsilon_{bb}(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \otimes p(t) \otimes h_{bb}(t) + n_{bb}(t)$

$V_{bb}(f) = \zeta(f+f_o)\,P(f)\,H_{bb}(f) + N_{bb}(f)$

After Deconvolution/Inverse Filter

$V(f) = \zeta(f+f_o)\text{rect}(\frac{f}{B})$

$\upsilon(t) = \left[\zeta(t)e^{-j2\pi f_o t}\right] \otimes B\frac{\sin(\pi Bt)}{(\pi Bt)}$

where$\frac{\sin(\pi Bt)}{(\pi Bt)} \equiv Sa(\pi Bt)$

POINT TARGET RESPONSE

$\upsilon_{RX}(t) = a_1 \upsilon_{TX}(t-\tau)$ where $\tau = \frac{2R}{c}$

$a_1 \propto \sqrt{\frac{G_t G_r \sigma \lambda^2}{(4\pi)^3 R^4}}$ (narrowband)

$\upsilon_{RX}(t) = \sum_{i=1}^{N} a_i \upsilon_{TX}(t-\tau_i)$ where $\tau_i = \frac{2R_i}{c}$

Baseband

$\upsilon_{bb}(t) = \upsilon_{RX}(t)\, e^{-j\omega_o t} \otimes h_{bb}(t) = \zeta\, p(t-\tau)\, e^{-j\omega_o \tau} \otimes h_{bb}(t)$

After deconvolution filtering

$\upsilon(t) = a_1 B\, Sa(\pi B[t-\tau]) e^{-j2\pi f_o \tau}$

$\psi = arg\left\{e^{-j2\pi f_o \tau}\right\} = arg\left\{e^{-j4\pi R/\lambda}\right\}$

Resolution

$\delta t_{3dB} \approx \frac{0.89}{B} \quad \delta R_{3dB} = \frac{c\delta t_{3dB}}{2} \approx \frac{c}{2B}(0.89)$

Radar Filters

Ideal Spectral Reconstruction (deconvolution/inverse) Filter

$H_{IRF}(f) = \frac{1}{P(f)H_{bb}(f)}$ over $-\frac{B}{2} \leq f \leq \frac{B}{2}$

Matched Filter (MF) $H_{MF}(f) = \frac{P^*(f)}{H_{bb}(f)} \approx P^*(f)$

Doppler Shift $f_D = \frac{-2\, dR/dt}{\lambda}$

MONOCHROME PULSE

RF: $\upsilon_{RF}(t) = rect\left(\frac{t}{T}\right) \cos(2\pi f_o t)$

Analytic: $\upsilon_{TX}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi f_o t}$

Baseband: $\upsilon_{bb}(t) = rect\left(\frac{t}{T}\right)$

Frequency Domain

$V_{TX}(f) = T\frac{\sin(\pi T(f-f_o))}{\pi T(f-f_o)}$

$V_{bb}(f) = T\frac{\sin(\pi T f)}{(\pi T f)}$

LINEAR FM CHIRP

RF signal $\upsilon_{RF}(t) = rect\left(\frac{t}{T}\right) \cos\left(2\pi[f_o t + \frac{1}{2}Kt^2]\right)$

Analytic: $\upsilon_{TX}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi[f_o t + \frac{1}{2}Kt^2]}$

Baseband: $\upsilon_{bb}(t) = rect\left(\frac{t}{T}\right) e^{j2\pi \frac{1}{2}Kt^2}$

Sweep range $\Delta f = KT$ [Hz]

Instantaneous Frequency

RF: $f_{RF}(t) = \frac{1}{2\pi}\frac{d\psi_{RF}(t)}{dt} = f_o + Kt \quad [Hz]$

Baseband: $f_{bb}(t) = Kt$ [Hz]

Dispersion factor $D = \Delta f\, T = KT^2$

Frequency Domain D¿50

$|\upsilon_{bb}(f)| \approx rect\left(\frac{f}{\Delta f}\right) \frac{1}{\sqrt{|K|}}$

$arg\{\upsilon_{bb}(f)\} = W\left\{-j\frac{\pi}{K}f^2\right\}$