EEE4114F: Digital Signal Processing

Class Test

19 March 2017

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.
- An information sheet is attached.

---

1. (5 marks) The input signal

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

is applied to a system with impulse response

$$h[n] = (1/3)^n u[n].$$

Find and plot the values of the output signal $y[n]$ over the range $n = -4$ to $n = 4$.

Since the input can be written as $x[n] = u[n] - u[n-10]$, the output will be

$$y[n] = h[n] * x[n] = h[n] * (u[n] - u[n-10]) = g[n] - g[n-10]$$

with $g[n] = h[n] * u[n]$.

Now

$$g[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k],$$

where the sum is truncated because $u[n-k] = 0$ for $n - k < 0$, or $k > n$. Since $h[k] = 0$ for $k < 0$ the output $g[n] = 0$ for $n < 0$. For $n > 0$ we have

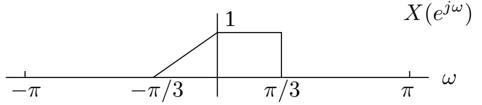
$$g[n] = \sum_{k=-\infty}^{n} (1/3)^k = \frac{1 - (1/3)^{n-1}}{1 - 1/3}.$$

The full step response is therefore

$$g[n] = \frac{3}{2}(1 - (1/3)^{n-1})u[n].$$

Since $g[n-10] = 0$ for $n < 10$ it has no effect on the output values over the range specified, so it is easy to find and plot the required values of $y[n]$.

2. (5 marks) A sequence $x[n]$ has a zero-phase DTFT $X(e^{j\omega})$ given below:



Sketch the magnitude and phase of the DTFT of the following sequences:

(a) $y_1 n] = x[n-1]$.

(b) $y_2[n] = x^*[-n]$.

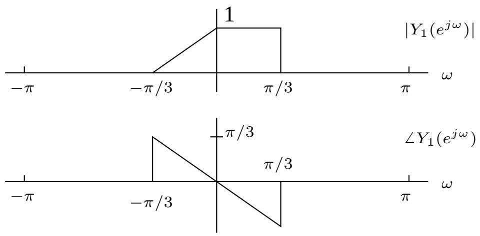
(a) The transform satisfies $Y_1(e^{j\omega}) = e^{-j\omega}X(e^{j\omega})$. Since $X(e^{j\omega})$ is zero phase we have $|Y_1(e^{j\omega})| = X(e^{j\omega})$ and $\angle Y_1(e^{j\omega}) = -\omega$, plotted below.

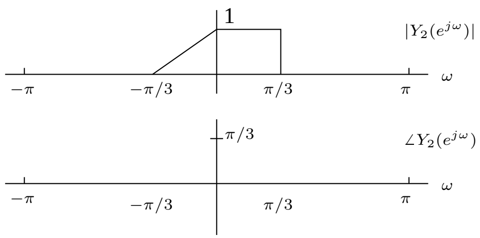
(b) The transform is as follows:

$$Y_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x^*[m]e^{j\omega m}$$

$$= \left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \right)^* = X^*(e^{j\omega}) = X(e^{j\omega}),$$

where the last step is true because $X(e^{j\omega})$ is real.





3. (5 marks) Consider the discrete LTI system represented by

$$y[n] = x[n] - x[n-1]$$

where $x[n]$ and $y[n]$ are the input and output respectively.

(a) Determine and plot the impulse response $h[n]$. Is the system stable?

(b) Determine and plot the step response corresponding to $x[n] = u[n]$.

(c) Find $H(e^{j\omega})$ and plot its magnitude.

(d) Determine and plot the response to the input $x[n] = (-1)^n$.

(a) The impulse response $h[n]$ is the output when the input is $x[n] = \delta[n]$, so $h[n] = \delta[n] - \delta[n-1]$. Since $\sum_{n=-\infty}^{\infty} |h[n]| = 2 < \infty$ the system is stable.

(b) When the input is $x[n] = u[n]$ then the output will be $y[n] = u[n] - u[n-1] = \delta[n]$.

(c) The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (\delta[n] - \delta[n-1])\, e^{-j\omega n} = 1 - e^{-j\omega} = e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})$$

$$= 2je^{-j\omega/2}\sin(\omega/2),$$

so $|H(e^{j\omega})| = 2|\sin(\omega/2)|$.

(d) The response to $x[n] = (-1)^n = e^{j\pi n}$ will be $y[n] = H(e^{j\pi})e^{j\pi n} = 2e^{j\pi n} = 2(-1)^n$.

4. (5 marks) A system has impulse response $h[n] = (1/2)^n u[n]$. Determine the input $x[n]$ to the system if the output is given by $y[n] = 2\delta[n-4]$. Explicitly state regions of convergence for any Z-transforms.

The system function is

$$H(z) = \frac{1}{1 - 1/2z^{-1}}$$

with ROC $|z| > 1/2$. The output is

$$Y(z) = 2z^{-4}$$

for all z. The input can then be found to be

$$X(z) = \frac{Y(z)}{H(z)} = 2z^{-4}(1 - 1/2z^{-1}) = 2z^{-4} - z^{-5}$$

with ROC the entire z-plane, so the input is

$$x[n] = 2\delta[n-4] - \delta[n-5].$$

Discrete-time Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common discrete-time Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Z-transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(z)$, $Y(z)$	ROC	Property
$ax[n] + by[n]$	$aX(z) + bY(z)$	ROC contains $R_x \cap R_y$	Linearity
$x[n - n_d]$	$z^{-n_d} X(z)$	ROC $= R_x$	Time shift
$z_0^n x[n]$	$X(z/z_0)$	ROC $= \lvert z_0\rvert R_x$	Frequency scale
$x^*[-n]$	$X^*(1/z^*)$	ROC $= \frac{1}{R_x}$	Time reversal
$nx[n]$	$-z\frac{dX(z)}{dz}$	ROC $= R_x$	Frequency diff.
$x[n] * y[n]$	$X(z)Y(z)$	ROC contains $R_x \cap R_y$	Convolution
$x^*[n]$	$X^*(z^*)$	ROC $= R_x$	Conjugation

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$