EEE4001F: Digital Signal Processing

Class Test 1

18 March 2009

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Find $w[n] = x[n] * y[n]$ with

$$x[n] = u[-n] \quad \text{and} \quad y[n] = 1.8^n u[-n].$$

There are many ways to do this: graphically, multiplying in the z-transform domain and inverting, or algebraically. This solution takes the latter approach.

These signals are both left-sided, which can be confusing. However, if we define time-reversed signals

$$w_r[n] = w[-n], \qquad x_r[n] = x[-n], \qquad y_r[n] = y[-n],$$

it is quite easy to see that $w[n] = x[n] * y[n]$ implies that $w_r[n] = x_r[n] * y_r[n]$:

$$\begin{aligned} w_r[n] = w[-n] &= \sum_{k=-\infty}^{\infty} x[k]y[-n-k] = \sum_{m=-\infty}^{\infty} x[-m]y[-n+m] \\ &= \sum_{m=-\infty}^{\infty} x[-m]y[-(n-m)] = \sum_{m=-\infty}^{\infty} x_r[m]y_r[n-m] = x_r[n] * y_r[n]. \end{aligned}$$

Now we convolve $x_r[n] = u[n]$ and $y_r[n] = (1.8)^{-n}u[n] = (1/1.8)^n u[n] = au[n]$ (with $|a| = |1/1.8| < 1$).

For $n < 0$ there is no overlap of $x_r[m]$ and $y_r[n-m]$, so $w_r[n] = 0$. For $n \geq 0$,

$$\begin{aligned} w_r[n] &= \sum_{k=-\infty}^{\infty} u[n-k]a^k u[k] = \sum_{k=0}^{\infty} u[n-k]a^k = \sum_{k=0}^{n} a^k = \sum_{k=0}^{(n+1)-1} a^k \\ &= \frac{1-a^{n+1}}{1-a}. \end{aligned}$$

Thus

$$w[n] = w_r[-n] = \begin{cases} 0 & n > 0 \\ \frac{1-a^{-n+1}}{1-a} & n \leq 0. \end{cases}$$

2. (5 marks) Consider the discrete-time linear time-invariant system described by the following impulse response:

$$h[n] = (2 - (0.2)^{n-1})u[n],$$

where $u[n]$ denotes the unit step function.

(a) Is the system stable? Why?

(b) Find the system function $H(z)$ and determine a linear constant coefficient difference equation that describes the system.

(a) As $n \to \infty$, $h[n] \to 2$. Since the system with the unit step as the impulse response is not stable, it is reasonable to suppose that the system given here is also unstable.

(b) Since

$$h[n] = 2u[n] - (0.2)^{n-1}u[n] = 2u[n] - (0.2)^{-1}(0.2)^n u[n] = 2u[n] - 5(0.2)^n u[n],$$

we have

$$H(z) = \frac{2}{1-z^{-1}} - \frac{5}{1-0.2z^{-1}} = \frac{2(1-0.2z^{-1}) - 5(1-z^{-1})}{(1-z^{-1})(1-0.2z^{-1})}$$
$$= \frac{2 - 0.4z^{-1} - 5 + 5z^{-1}}{1 - 0.2z^{-1} - z^{-1} + 0.2z^{-2}} = \frac{-3 + 4.6z^{-1}}{1 - 1.2z^{-1} + 0.2z^{-2}}.$$

The region of convergence is $|z| > 1$. Since $Y(z) = H(z)X(z)$ we can multiply out and invert to give

$$y[n] - 1.2y[n-1] + 0.2y[n-2] = -3x[n] + 4.6x[n-1],$$

which can also be used to implement the system given the right set of initial conditions.

3. (5 marks) Consider a discrete-time LTI system with impulse response

$$h[n] = j^n u[n].$$

(a) Is the system BIBO stable? Substantiate.

(b) Find the system function $H(z)$ of this system and draw the pole-zero diagram. Note the z-transform property which states that if $x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$ with ROC R_x, then $z_o^n x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z/z_0)$ with ROC $|z_0|R_x$.

(c) Write a difference equation for the LTI system having the above impulse response.

(a) Since $j = e^{j\pi/2}$, the absolute sum of impulse response is

$$S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |(e^{j\pi/2})^n u[n]| = \sum_{n=0}^{\infty} |e^{j\pi/2}|^n = \sum_{n=0}^{\infty} 1.$$

This sum is not finite, so the system is not BIBO stable.

(b) Using the given property with $z_0 = e^{j\pi/2}$ on the pair $u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}$ for $|z| > 1$ gives

$$j^n u[n] = e^{j\frac{\pi}{2}n} u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - e^{j\frac{\pi}{2}} z^{-1}} = \frac{1}{1 - jz^{-1}} = \frac{z}{z - j}$$

for ROC $|z| > 1$, so there is a pole at $z = j$ and a a zero at the origin.

(c) Since

$$H(z) = \frac{1}{1 - jz^{-1}} = \frac{Y(z)}{X(z)}$$

we have $X(z) = Y(z)(1 - jz^{-1})$. Inverting the transform gives the required difference equation $x[n] = y[n] - jy[n-1]$.

4. (5 marks) Let $X(e^{j\omega})$ denote the DTFT of the sequence

$$x[n] = 2\delta[n+2] + 3\delta[n+1] - \delta[n] - 4\delta[n-2] + 3\delta[n-3].$$

Evaluate the following without computing the transform itself:

(a) $X(e^{j0})$.

(b) $X(e^{j\pi})$.

(c) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$ (bonus question).

(a) Since $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, we have

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]e^{j0n} = \sum_{n=-\infty}^{\infty} x[n] = 3 - 4 - 1 + 3 + 2 = 3.$$

(b) Similarly

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\pi n} = \sum_{n=-\infty}^{\infty} x[n](-1)^n = -3 - 4 - 1 - 3 + 2 = -9.$$

(c) If $v[n] = x[n]x[n]$, then the modulation property states that

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})X(e^{j(\omega-\theta)})d\theta,$$

so

$$V(e^{j0}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})X(e^{-j\theta})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})X^{\star}(e^{j\theta})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\theta})|^2 d\theta$$

But as seen before

$$V(e^{j0}) = \sum_{n=-\infty}^{\infty} v[n] = \sum_{n=-\infty}^{\infty} x[n]x[n] = 9 + 16 + 1 + 9 + 4 = 39,$$

so $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 39(2\pi)$.

Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n-n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$