EEE4001F: Digital Signal Processing

Class Test 2

30 April 2008

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Find $w[n] = x[n] * y[n]$ with

$$x[n] = (-1)^n \quad \text{and} \quad y[n] = \frac{\sin(\pi n/3)}{\pi n}.$$

We can think of $w[n]$ as the output of a system with impulse response $y[n]$ to the input signal $x[n]$. From the tables of Fourier transform pairs, the frequency response of the system is

$$Y(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \pi/3 \\ 0 & \pi/3 < |\omega| < \pi. \end{cases}$$

Since $x(t) = (-1)^n = e^{j\pi n}$ is a complex exponential of frequency $\omega = \pi$, the output is

$$w[n] = Y(e^{j\pi})e^{j\pi n} = 0.$$

(Alternatively find and sketch $X(e^{j\omega})$ and $Y(e^{j\omega})$, and show that the product $W(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$ is zero.)

2. (5 marks) Consider the following discrete-time signal:

$$x[n] = \begin{cases} \sin(\frac{n\pi}{4}) & \text{when } n/2 \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

Calculate and sketch the 8-point DFT (magnitude and phase) of the first 8 samples of $x[n]$, i.e. $x[0], x[1], \ldots, x[7]$. Show and motivate your calculations.

The only nonzero samples of $x[n]$ over the range 0 to 7 are $x[2] = 1$ and $x[6] = -1$. The DFT is therefore

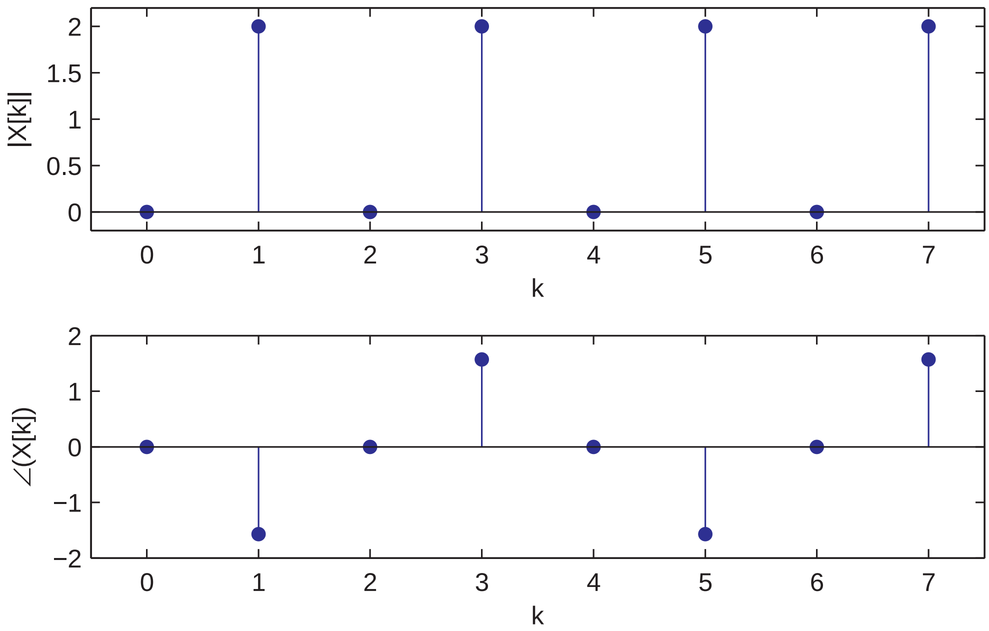
$$\begin{aligned} X[k] &= \sum_{n=0}^{7} x[n] W_8^{kn} = \sum_{n=0}^{7} x[n] e^{-j(\frac{2\pi}{8})kn} \\ &= e^{-j(\frac{2\pi}{8})2k} - e^{-j(\frac{2\pi}{8})6k} = e^{-j(\frac{\pi}{2})k} - e^{j(\frac{\pi}{2})k} \\ &= -\frac{2j}{2j}(e^{j(\frac{\pi}{2})k} - e^{-j(\frac{\pi}{2})k}) = -2j \sin(\pi k/2). \end{aligned}$$

Therefore

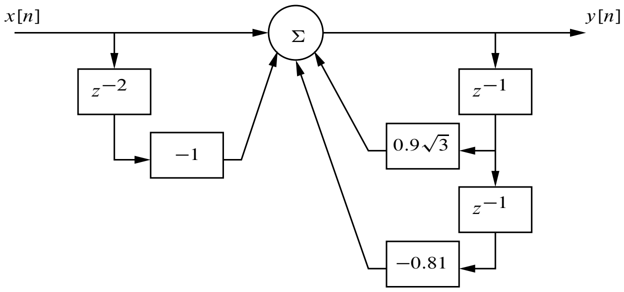
$$X[0] = X[4] = 0 \quad X[1] = X[5] = -2j = 2e^{-j\pi/2}$$

$$X[2] = X[6] = 0 \quad X[3] = X[7] = 2j = 2e^{j\pi/2}.$$

Magnitude and phase plots are as follows:



3. (5 marks) Consider the following LTI system:



(a) Show that the system function is

$$H(z) = \frac{(z-1)(z+1)}{(z-0.9e^{j\pi/6})(z-0.9e^{-j\pi/6})}.$$

(b) Sketch the magnitude frequency response $|H(e^{j\omega})|$ over the range $0 \leq \omega \leq 2\pi$. Indicate calculated amplitudes at $\omega = 0$, $\omega = \frac{\pi}{6}$, and $\omega = \pi$.

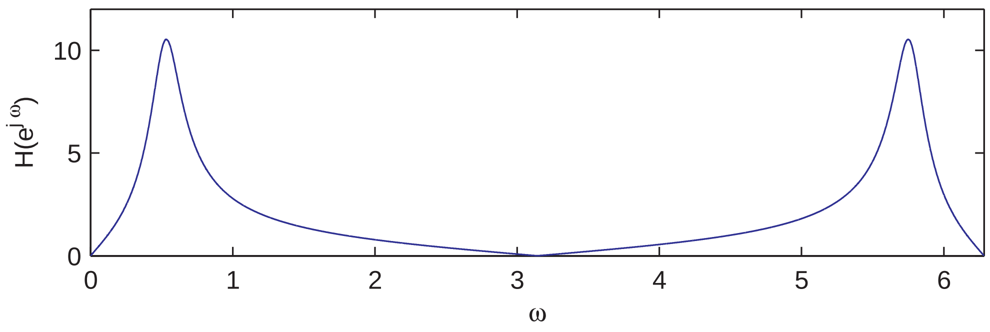
The difference equation for the system is

$$y[n] = 0.9\sqrt{(3)}y[n-1] - 0.81y[n-2] + x[n] - x[n-2].$$

By taking z-transforms, the system function is found to be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-2}}{1-0.9\sqrt{3}z^{-1}+0.81z^{-2}} = \frac{z^2-1}{(z-0.45(\sqrt{3}+j))(z-0.45(\sqrt{3}-j))}$$
$$= \frac{(z-1)(z+1)}{(z-0.45(2e^{j\pi/6}))(z-0.45(2e^{-j\pi/6}))} = \frac{(z-1)(z+1)}{(z-0.9e^{j\pi/6})(z-0.9e^{-j\pi/6})}.$$

which has zeros at $z = \pm 1$ and poles at $z = 0.9e^{\pm j\pi/6}$. Graphical methods can be used to find the magnitude response:



Specifically, $H(e^{j0}) = H(e^{j\pi}) = 0$, and $H(e^{j\pi/6}) \approx 10$.

4. (5 marks) One of the simplest filters is a backward-difference system, where

$$y[n] = x[n] - x[n-1].$$

Using sketches, notes, and equations, justify why this is a highpass filter.

The z-transform of the time-domain representation is
$Y(z) = X(z) - z^{-1}X(z) = X(z)(1 - z^{-1})$, so the system function is

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}.$$

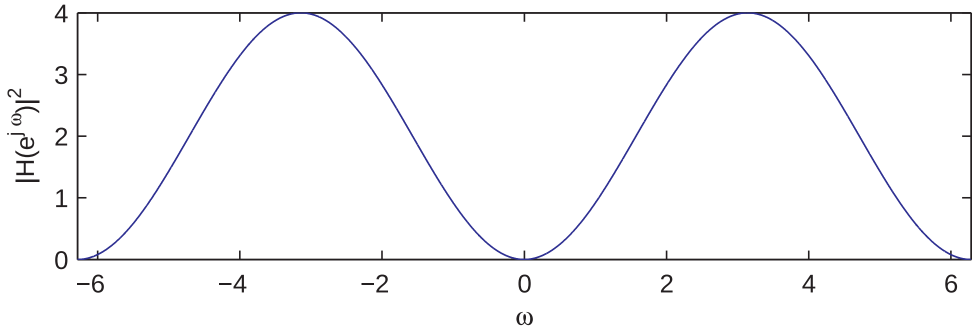
The frequency response is

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = 1 - e^{-j\omega},$$

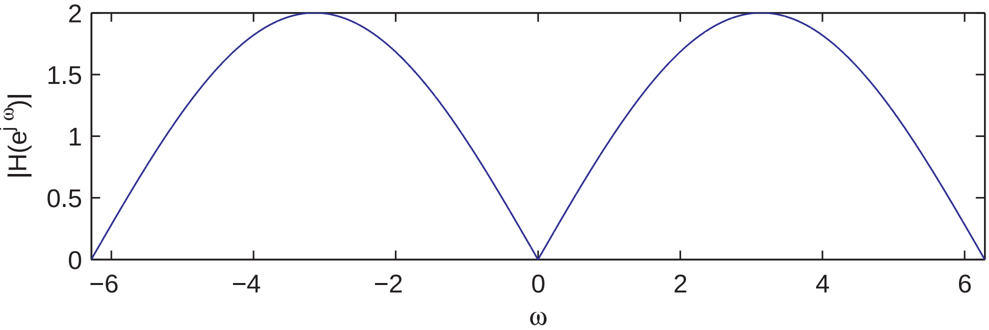
so the squared magnitude response is

$$\begin{aligned}|H(e^{j\omega})|^2 &= (1 - e^{-j\omega})(1 - e^{j\omega}) = 1 - e^{j\omega} - e^{-j\omega} + 1 \\ &= 2 - \frac{2}{2}(e^{j\omega} + e^{-j\omega}) = 2 - 2\cos(\omega).\end{aligned}$$

Thus we have a highpass filter characteristic:



or



Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{-j\omega})Y(e^{-j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$