EEE4001F: Digital Signal Processing

Class Test 2

15 May 2006

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Consider the z-transform

$$G(z) = \frac{(z^2 + 0.2z + 0.1)(z^2 - z + 0.5)}{(z^2 + 0.3z - 0.18)(z^2 - 2z + 4)}.$$

What are the possible regions of convergence for this transform? Discuss the type of inverse z-transform (left-sided, right-sided, two-sided, stable, unstable, etc.) associated with each of the ROCs. It is not necessary to compute the exact inverse transform.

By factoring the denominator polynomial we see that the four poles are at $-0.6, 0.3, 1 \pm j\sqrt{3}$. Since $|1 \pm j\sqrt{3}| = 2$, there are four possible regions of convergence: $|z| < 0.3$, $0.3 < |z| < 0.6$, $0.6 < |z| < 2$, and $|z| > 2$. For each ROC the inverse transforms have the following properties:

(a) $z < 0.3$: ROC inside innermost pole and does not include unit circle, so left-sided unstable inverse.

(b) $0.3 < |z| < 0.6$: ROC neither inside nor outside and does not include unit circle, so two-sided unstable inverse.

(c) $0.6 < |z| < 2$: ROC neither inside nor outside but does include unit circle, so two-sided stable inverse.

(d) $|z| > 2$: ROC outside outermost pole and does not include unit circle, so right-sided unstable inverse.

2. (5 marks) Determine the z-transforms of the following sequences and their respective ROCs:

   (a) $x_1[n] = \alpha^n u[n-2]$

   (b) $x_2[n] = \alpha^n u[-n-3]$.

(a) The z-transform is

$$X_1(z) = \sum n = -\infty^{\infty} \alpha^n u[n-2] z^{-n} = \sum_{n=2}^{\infty} \alpha^n z^{-n} = \sum_{n=2}^{\infty} (\alpha z^{-1})^n$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n - (\alpha z^{-1})^0 - (\alpha z^{-1})^1 = \frac{1}{1-\alpha z^{-1}} - (1 + \alpha z^{-1})$$

as long as $|\alpha z^{-1}| < 1$ (so the ROC is $|z| > |\alpha|$). This can be simplified to give the required transform

$$X_1(z) = \frac{\alpha^2 z^{-2}}{1-\alpha z^{-1}} \qquad |z| > |\alpha|.$$

(b) The z-transform is

$$X_1(z) = \sum n = -\infty^{\infty} \alpha^n u[-n-3] z^{-n} = \sum m = -\infty^{\infty} \alpha^{-m} u[m-3] z^m$$

$$= \sum_{m=3}^{\infty} \alpha^{-m} z^m = \sum_{m=0}^{\infty} (\alpha^{-1} z)^m - \left[1 + (\alpha^{-1} z) + (\alpha^{-1} z)^2\right]$$

$$= \frac{1}{1-\alpha^{-1} z} - \left[1 + (\alpha^{-1} z) + (\alpha^{-1} z)^2\right]$$

with the ROC is $|z| < |\alpha|$). Simplify if required.

3. (5 marks) Suppose $X[k]$ is the N-point DFT of $x[n]$. If $x[n]$ is real, what symmetry does this imply on the elements of $X[k]$? Recall that if a is a real number, then $a = a^\star$.

Taking the conjugate of the DFT relation

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

and applying $x[n] = x^*[n]$ we get

$$\begin{aligned} X^*[k] &= \sum_{n=0}^{N-1} x^*[n] e^{j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}Nn} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(N-k)n} = X[n-k]. \end{aligned}$$

Thus the sequence $X[k]$ is conjugate symmetric.

4. (5 marks) Determine an expression for the frequency response $H(e^{j\omega})$ of a causal LTI discrete-time system characterised by the input-output relation

$$y[n] = x[n] + \alpha y[n-R], \qquad |\alpha| < 1,$$

where $x[n]$ is the input and $y[n]$ the output to the system. How many peaks and dips of the magnitude response occur in the range $0 \leq \omega < \pi$, and what are their locations?

Taking the z-transform of the expression we arrive at the system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \alpha z^{-R}}.$$

This has poles whenever $1 - \alpha z^{-R} = 0$, or when $z^R = \alpha$. This is a polynomial of order R, and has R roots uniformly spaced around the circle in the z-plane with radius $\alpha^{1/R}$ — thus the poles are at $z = \alpha^{1/R} e^{j(2\pi/R)k}$ for $k = 0, 1, \ldots, R-1$.

Putting $z = e^{j\omega}$ and drawing the poles in the z-plane for some values of R we readily see that the peaks in the transfer function will be at angles $\omega = (2\pi/R)k$ for integer k, and that the dips will be halfway in between these. For example, the case of $R = 6$ is shown below: