EEE401F: Digital Signal Processing

Class Test 1

18 March 2004

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *five* questions, totalling 25 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Find and sketch the unit step response of the causal LTI processor defined by the following recurrence formula:

$$y[n] = -0.5y[n-1] + x[n]$$

Is the resulting sequence stable?

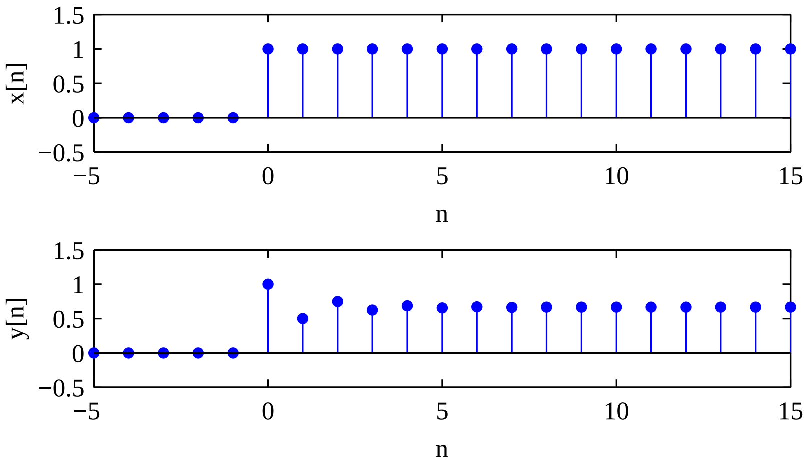
With input $x[n] = u[n]$, we must have $y[-1] = y[-2] = \cdots = 0$ since the system is causal. Thus

$$y[n] = -0.5y[n-1] + u[n]$$

yields

$$\begin{aligned} y[0] &= u[0] = 1 \\ y[1] &= -\frac{1}{2} + \frac{2}{2} = \frac{1}{2} \\ y[3] &= -\frac{1}{4} + \frac{4}{4} = \frac{3}{4} \\ y[4] &= -\frac{3}{8} + \frac{8}{8} = \frac{5}{8} \\ y[5] &= -\frac{5}{16} + \frac{16}{16} = \frac{11}{16} \end{aligned}$$

and so on:



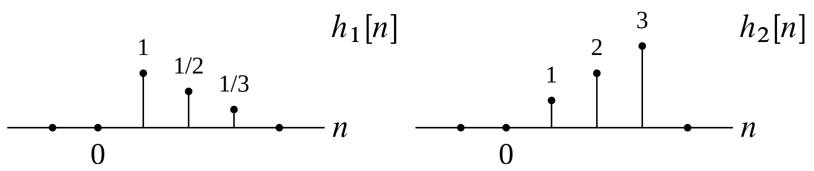
2. (5 marks) Find the impulse response of an overall system formed by cascading two LTI processors with the impulse responses:

$$h_1[n] = \begin{cases} \frac{1}{n} & (0 < n < 4) \\ 0 & \text{otherwise} \end{cases}$$

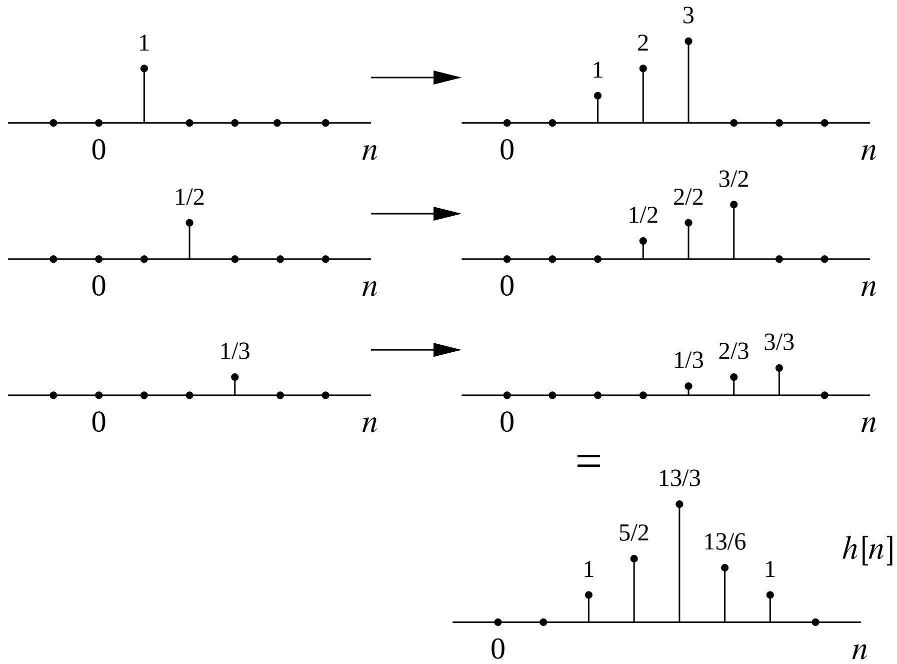
and

$$h_2[n] = \begin{cases} n & (0 < n < 4) \\ 0 & \text{otherwise.} \end{cases}$$

The signals are



so the impulse response of the overall system is the convolution of the two:



3. (5 marks) Find a nonrecursive recurrence formula which, from the DSP point of view, is equivalent to the following recursive formula for a causal filter:

$$y[n] = y[n-1] + x[n] - x[n-7].$$

What is the relative computational economy of the recursive and nonrecirsive versions?

Taking the z-transform of $y[n] = y[n-1] + x[n] - x[n-7]$ we get

$$Y(z)(1 - z^{-1}) = X(z)(1 - z^{-7}),$$

so

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - z^{-7})}{(1 - z^{-1})} = \frac{(z^7 - 1)}{z^6(z - 1)} = \frac{1}{(1 - z^{-1})} - \frac{z^{-7}}{(1 - z^{-1})}.$$

The system has 6 poles at $z = 0$, and 1 pole at $z - 1$. Since it is causal, thr ROC must be outside the outermost pole: $|z| > 1$. Inverting the partial fraction expansion above under this condition yields $h[n] = u[n] - u[n-7]$. This is the nonrecursive specification of the filter.

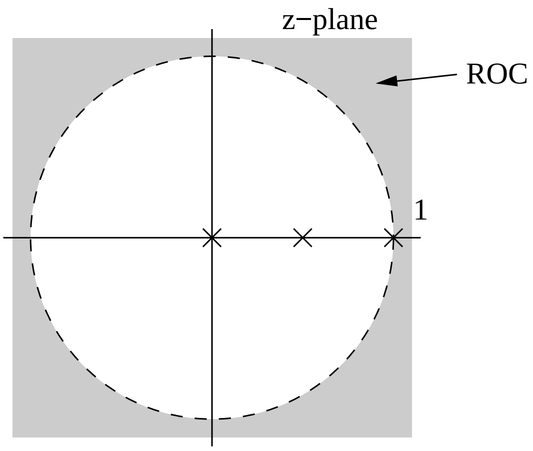
The nonrecursive form for the filter requires 6 additions per output sample (since the seven multiplications are by unity). The recursive form requires 2 additions.

4. (5 marks) A signal has the z-transform

$$X(z) = \frac{1}{z(z-1)(2z-1)},$$

with region of convergence $|z| > 1$. Draw a pole-zero plot of the signal in the z-plane, and use the method of partial fractions to recover the signal $x[n]$. Is the signal stable? Is the signal causal?

In the z-plane the signal is



Therefore the signal is causal (ROC outside outermost pole) but not stable (ROC does not contain the unit circle).

The z-transform can be written as

$$H(z) = \frac{\frac{1}{2}z^{-3}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{1}{2}z^{-3}\left[\frac{a}{(1-z^{-1})} + \frac{b}{(1-\frac{1}{2}z^{-1})}\right],$$

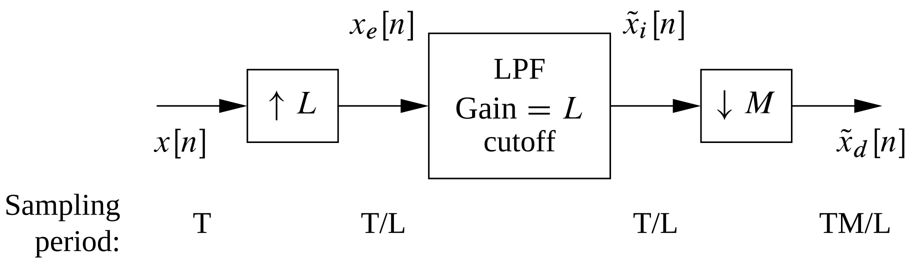
where it is easily shown that $a = 2$ and $b = -1$. The term in square brackets inverts quite simply to

$$x_1[n] = 2u[n] - \left(\frac{1}{2}\right)^n u[n],$$

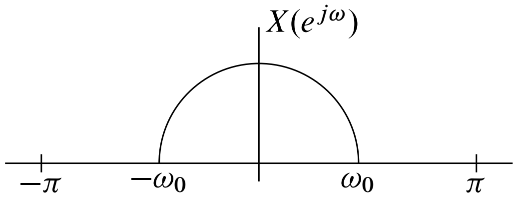
and the inverse is this quantity delayed by 3 samples and scaled by $1/2$:

$$\begin{aligned} x[n] &= \frac{1}{2}(2u[n-3] - (1/2)^{n-3}u[n-3]) \\ &= u[n-3] - (1/2)^{n-2}u[n-3]. \end{aligned}$$

5. (5 marks) Consider the system below:

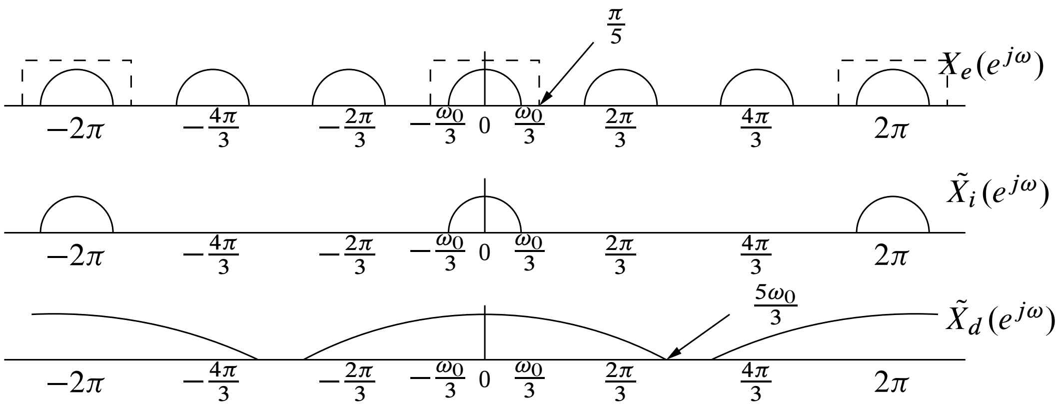


where the cutoff of the LPF is at $\min(\pi/L, \pi/M)$. The Fourier transform of the input signal $x[n]$ is



For $M = 5$ and $L = 3$, draw the transforms of the signals at each stage, and specify the maximum value of ω_0 such that $\tilde{X}_d(e^{j\omega}) = aX(e^{jM\omega/L})$ for some a.

The signals are as follows:



At the output of the lowpass filter, we have no loss of data (truncation) as long as

$$\frac{\omega_0}{3} < \frac{\pi}{5}.$$

This is the condition under which $\tilde{X}_d(e^{j\omega})$ is just a stretched-out replica of $X(e^{j\omega})$. The maximum value of ω_o is therefore $3\pi/5$.