EEE401F Class Test

30 April 2003

Name:

Student number:

Information

- The test is closed-book.
- This test has *six* questions, totalling 30 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Consider the system function

$$H(z) = \frac{z-1}{(z-2)^2}$$

for a *causal* system. Draw a pole-zero diagram for the system, and indicate the region of convergence. Is the system stable? Why?

- Single zero at $z = 1$ since $H(z)|_{z=1} = 0$
- Double pole at $z = 2$ since $H(z)|_{z=2} = \infty$
- The system is causal, so the ROC is outside of the outermost pole: —z—¿2
- The system is stable if the ROC includes the unit circle, which it does not. Hence the system is unstable.

2. (5 marks) Consider the system function

$$H(z) = \frac{z-1}{(z-2)^2}$$

for a *causal* system. Using the power series method, find the first five values of the causal discrete-time impulse response of the system.

Start with long division of $H(z)$ to get the required power series expansion:

$$\begin{array}{r|llllll}
 & z^{-1} & +3z^{-2} & +8z^{-3} & +20z^{-4} & +\cdots \\
\hline
z^2-4z+4 & z & -1 & & & \\
 & z & -4 & +4z^{-1} & & \\
\cline{2-4}
 & & 3 & -4z^{-1} & & \\
 & & 3 & -12z^{-1} & +12z^{-2} & \\
\cline{3-5}
 & & & 8z^{-1} & -12z^{-2} & \\
 & & & 8z^{-1} & -32z^{-2} & +32z^{-3} \\
\cline{4-6}
 & & & & 20z^{-2} & -32z^{-3}
\end{array}$$

Now, since

$$\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} \\
&= \ldots + h[-2]z^2 + h[-1]z^1 + h[0] + h[1]z^{-1} + h[2]z^{-2} + \ldots,
\end{aligned}$$

we can simply equate coefficients to get the impulse response values

$$\begin{aligned}
h[n] &= 0 \quad \text{for} \quad n < 0 \\
h[0] &= 0 \\
h[1] &= 1 \\
h[2] &= 3 \\
h[3] &= 8 \\
h[4] &= 20 \\
h[5] &= \ldots
\end{aligned}$$

3. (6 marks) Consider the system function

$$H(z) = \frac{z-1}{(z-2)^2}$$

for a *causal* system. Using the z-transform pair

$$na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|$$

and the property

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z),$$

find a closed-form expression for the impulse response $h[n]$.

Write the system function in terms of quantities that we know the inverse transform of:

$$\begin{aligned} H(z) &= \frac{z^{-2}(z-1)}{[z^{-1}(z-2)]^2} = \frac{z^{-1} - z^{-2}}{(1-2z^{-1})^2} \\ &= \frac{1}{2}\frac{2z^{-1}}{(1-2z^{-1})^2} - \frac{z^{-1}}{2}\frac{2z^{-1}}{(1-2z^{-1})^2}, \end{aligned}$$

so

$$h[n] = \frac{1}{2} n 2^n u[n] - \frac{1}{2}(n-1)2^(n-1)u[n-1].$$

Check by evaluating the first 5 points:

$$\begin{aligned} h[0] &= \frac{1}{2}(0) = 0 \\ h[1] &= \frac{1}{2}(2) - 0 = 1 \\ h[2] &= \frac{1}{2}(2)(4) - \frac{1}{2}(2) = 4 - 1 = 3 \\ h[3] &= \frac{1}{2}(3)(2^3) - \frac{1}{2}(2)(2^2) = 12 - 4 = 8 \\ h[4] &= \dots. \end{aligned}$$

4. (4 marks) Consider the system function

$$H(z) = \frac{z-1}{(z-2)^2}$$

for a *causal* system. Find a difference equation (relating the input $x[n]$ to the output $y[n]$) for the system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z-1}{z^2 - 4z + 4},$$

so

$$z^2 Y(z) - 4zY(z) + 4Y(z) = zX(z) - X(z).$$

Inverting gives

$$y[n+2] - 4y[n+1] + 4y[n] = x[n+1] - x[n].$$

Alternatively,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - z^{-2}}{1 - 4z^{-1} + 4z^{-2}},$$

so

$$Y(z) - 4z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z).$$

Inverting gives

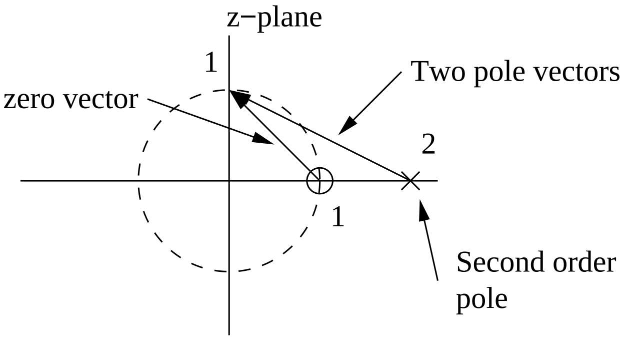
$$y[n] - 4y[n-1] + 4y[n-2] = x[n-1] - x[n-2].$$

5. (6 marks) Consider the system function

$$H(z) = \frac{z-1}{(z-2)^2}$$

for a *causal* system. Use graphical arguments to find the magnitude and phase of the frequency response of the system for $\omega = \pi/2$ and $\omega = \pi$.

For $\omega = \pi/2$:

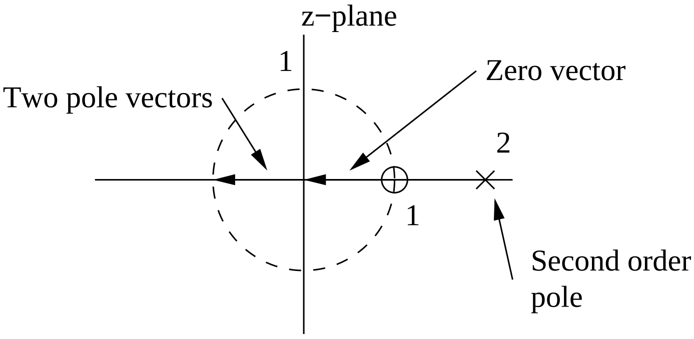


$$\text{Gain} = \frac{\prod \text{length of zero vectors}}{\prod \text{length of pole vectors}} = \frac{\sqrt{1^2+1^2}}{\sqrt{1^2+2^2}\sqrt{1^2+2^2}} = \frac{\sqrt{2}}{5}.$$

$$\text{Phase} = \sum \text{angle of zero vectors} - \sum \text{angle of pole vectors}$$

$$= \frac{3\pi}{4} - 2(\pi - \tan^{-1}(1/2)) = \frac{3\pi}{4} - 2(2.6779) = -2.9997 \text{ rad} = -171.87^\circ = 188.13^\circ.$$

For $\omega = \pi$:



$$\text{Gain} = \frac{\prod \text{length of zero vectors}}{\prod \text{length of pole vectors}} = \frac{2}{(3)(3)} = \frac{2}{9}.$$

$$\text{Phase} = \sum \text{angle of zero vectors} - \sum \text{angle of pole vectors}$$

$$= \pi - 2(\pi) = -\pi \text{ rad} = \pi \text{ rad} = 180^\circ.$$

6. (4 marks) Consider the system function

$$H(z) = \frac{z-1}{(z-2)^2}$$

for a *causal* system. Use $H(z)$ directly (algebraically) to find the magnitude and phase of the frequency response of the system for $\omega = \pi/2$.

We want to know the magnitude and phase of

$$H(e^{j\pi/2}) = \left.\frac{z-1}{(z-2)^2}\right|_{z=e^{j\pi/2}} = \frac{e^{j\pi/2}-1}{(e^{j\pi/2}-2)^2} = \frac{j-1}{(j-2)^2}$$

(since $e^{j\pi/2} = j$). Simplifying gives

$$\begin{aligned} H(e^{j\pi/2}) &= \frac{i-1}{i^2-4i+4} = \frac{i-1}{3-4i} = \frac{i-1}{3-4i}\frac{3+4i}{3+4i} \\ &= \frac{3i-4-3-4i}{9+16} = \frac{-7-j}{25}. \end{aligned}$$

The magnitude of the response is therefore

$$|H(e^{j\pi/2})| = \left|\frac{-7-j}{25}\right| = \frac{\sqrt{49+1}}{25} = \frac{\sqrt{25(2)}}{5(5)} = \frac{\sqrt{2}}{5}.$$

The phase is the angle of this vector, which lies in the third quanrant of the plane. The principal angle is therefore

$$\theta_p = \tan^{-1}\left(\frac{1}{7}\right) = 8.13°,$$

and the overall phase is $180° + \theta_p = 188.13°$.