EEE4114F: Digital Signal Processing

Class Test 2

11 April 2019

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 60 marks.
- Answer *all* the questions.
- You have 60 minutes.

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1. (15 marks) Consider the right-sided signal $x[n]$ described by the following z-transform:

$$X(z) = \frac{-4 + 10z^{-1} - 5z^{-2}}{1 - 2z^{-1} + z^{-2}}.$$

Determine a real closed-form expression for $x[n]$ and sketch it for $n = -3, \ldots, 3$.

The transform can be factored as

$$X(z) = \frac{-4 + 10z^{-1} - 5z^{-2}}{(1 - z^{-1})^2}.$$

This has a pole of order 2 at $z = 1$, so the ROC for a right sided signal will be $|z| > 1$. The transform pair

$$na^n u[n] \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{az^{-1}}{(1 - az^{-1})^2} \qquad |z| > |a|$$

for $a = 1$ gives

$$nu[n] \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z^{-1}}{(1 - az^{-1})^2} \qquad |z| > 1.$$

Define $h[n] = nu[n]$. Since we can write

$$X(z) = -4z\frac{z^{-1}}{(1 - z^{-1})^2} + 10\frac{z^{-1}}{(1 - z^{-1})^2} + -5z^{-1}\frac{z^{-1}}{(1 - z^{-1})^2}$$

we can use the time shift property to find the require inverse:

$$x[n] = -4h[n+1] + 10h[n] - 5h[n-1] = -4(n+1)u[n+1] + 10nu[n] - 5(n-1)u[n-1].$$

We can use this expression to find the required values. For $n < -1$ we have $x[n] = 0$ since all the unit step functions are zero. For $n >= 1$ all the unit step functions are one and $x[n] = -4(n+1) + 10n - 5(n-1) = n + 1$. Finally $x[-1] = -4(-1+1) = 0$ and $x[0] = -4(0+1) + 10(0) = -4$, and the required plot follows.

2. (15 marks) Suppose the 8-point DFT of $\{x[0], x[1], x[2], x[3], 0, 0, 0, 0\}$ is given by the elements of the vector below:

$$\mathbf{X}_8 = \begin{pmatrix} 2 \\ (1/2 - \sqrt{3/2}) - j3/2 \\ -1 + j\sqrt{3} \\ (1/2 + \sqrt{3/2}) + j3/2 \\ 2 \\ (1/2 + \sqrt{3/2}) - j3/2 \\ -1 - j\sqrt{3} \\ (1/2 - \sqrt{3/2}) + j3/2 \end{pmatrix}.$$

(a) What is the 4-point DFT of $\{x[0], x[1], x[2], x[3]\}$?

(b) Find the value of $x[0]$.

(a) The 8-point DFT in this case satisfies

$$X_8[k] = \sum_{n=0}^{7} x[n] W_8^{kn} = \sum_{n=0}^{3} x[n] W_8^{kn} = \sum_{n=0}^{3} x[n] (e^{-j2\pi/8})^{kn}$$

The 4-point DFT is

$$X_4[k] = \sum_{n=0}^{3} x[n] W_4^{kn} = \sum_{n=0}^{3} x[n] (e^{-j2\pi/4})^{kn} = \sum_{n=0}^{3} x[n] (e^{-j2\pi/8})^{(2k)n} = X_8[2k],$$

so the required DFT is

$$\mathbf{X}_4 = \begin{pmatrix} 2 & -1 + j\sqrt{3} & 2 & -1 - j\sqrt{3} \end{pmatrix}.$$

(b) The required value is

$$\begin{aligned} x[0] &= \frac{1}{8} \sum_{k=0}^{8} X[k] W_8^{-k0} = \frac{1}{8} \sum_{k=0}^{8} X[k] \\ &= \frac{1}{8}(2 + 2(1/2 - \sqrt{3/2}) + 2(-1) + 2(1/2 + \sqrt{3/2}) + 2) = 1/2. \end{aligned}$$

3. (15 marks) We have collected n data points (all positive) from a data source that follows an exponential distribution. These collected data are known to be IID (independent and identically distributed).

For exponential distribution, the pdf for observation x (given $x \geq 0$) is given by $p(x) = \lambda \exp(-\lambda x)$. Find the maximum likelihood estimate (MLE) for the parameter λ. (Hint: remember how we found the ML estiimate of the mean for a Gaussian distribution.)

Suggested procedure: find the likelihood probability of the parameter λ given the n observations (5 marks); then find the log-likelihood (4 marks); next find the value of the parameter which will maximise the log-likelihood (6 marks).

The students should roughly show the results as shown in the link https://www.statlect.com/fundamentals-of-statistics/exponential-distribution-maximum-likelihood.

4. (15 marks) Imagine that machines have gone really intelligent and somehow destroyed all humans. Then a machine discovers two thermometers, one measuring temperature in Celsius and one in Fahrenheit. The machine is confused as it does not know the relation that links these two measurements, so it gathers a set of 7 measurements:

$$-40, -10, 0, 8, 15, 22, 38$$

in Celsius corresponding to

$$-40, 14, 32, 46, 59, 72, 100$$

in Fahrenheit. It builds an ANN model to convert Celsius value to Fahrenheit using the following:

```
l0 = tf.keras.layers.Dense(units=50, activation='relu', input_shape=[1])
l1 = tf.keras.layers.Dense(units=1)
model = tf.keras.Sequential([l0, l1])
```

When tested with new measurements this model does not perform well, giving an error of around 2–3 degrees! Can you explain why?

Suggested procedure: first draw a rough plot of the data available (4 marks); comment on the complexity of the task (2 marks); now look at the code and explain what each layer means (4 marks); then comment on what might be going wrong (5 marks).

Observe that the problem at hand is linear regression. However, the code is complicated in two ways: it uses way too many links (i.e. parameters) and also adds a nonlinear activation `relu`. This causes the shortage of data (in that we just have seven data points and 51 weights to learn) and that there is a forced non-linearity. Hence, the model (though sophisticated) is too complicated for the problem. This also shows that an over-complicated model mostly tends to cause bad performance as suggested by the law of parsimony.

Discrete-time Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common discrete-time Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$

Z-transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(z)$, $Y(z)$	ROC	Property
$ax[n] + by[n]$	$aX(z) + bY(z)$	ROC contains $R_x \cap R_y$	Linearity
$x[n - n_d]$	$z^{-n_d} X(z)$	ROC $= R_x$	Time shift
$z_0^n x[n]$	$X(z/z_0)$	ROC $= \lvert z_0\rvert R_x$	Frequency scale
$x^*[-n]$	$X^*(1/z^*)$	ROC $= \frac{1}{R_x}$	Time reversal
$nx[n]$	$-z\frac{dX(z)}{dz}$	ROC $= R_x$	Frequency diff.
$x[n] * y[n]$	$X(z)Y(z)$	ROC contains $R_x \cap R_y$	Convolution
$x^*[n]$	$X^*(z^*)$	ROC $= R_x$	Conjugation

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n - m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$