EEE4114F: Digital Signal Processing

Class Test

7 March 2019

SOLUTIONS

Name:

Student number:

Information

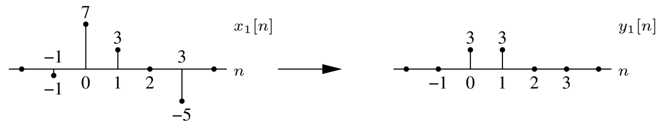
- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 60 minutes.
- An information sheet is attached.

---

1. (5 marks) The output $y[n]$ of a median filter is given by

$$y[n] = \mathrm{med}(x[n-1], x[n], x[n+1]).$$

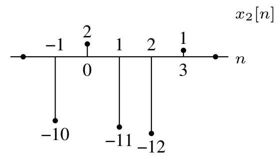
Here the median function, med( ), lists the 3 samples in descending order and selects the value in the middle of the list. For example, the input $x_1[n]$ generates the output $y_1[n]$ below:



(a) Is the system causal? Why?

(b) Is the system stable? Why?

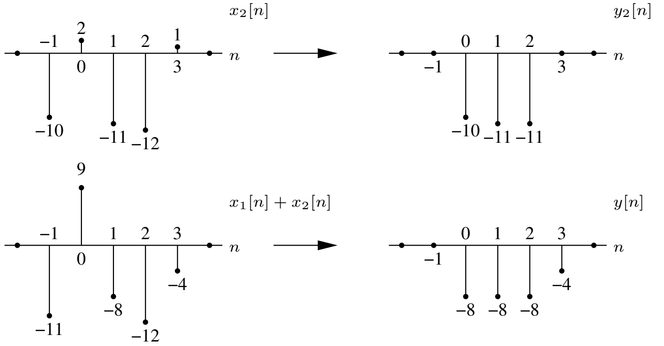
(c) Comment on whether the above median filter is linear. Justify your comments and construct a simple relevant example to illustrate your conclusion. For this purpose you may want to use the example pair given above and the input sequence below:



(a) To calculate the output at say $n = 10$ we need to know the inputs $x[9]$, $x[10]$, and $x[11]$. Since this last input is in the future the system is not causal.

(b) The output at any instant is always generated by taking the value of a selected input sample, so every value in $y[n]$ must occur somewhere in $x[n]$. Thus if $|x[n]| \leq B_x$ for some B_x then we must also have $|y[n]| \leq B_x$. Since a bounded input always produces a bounded output (with the same bound), the system is stable.

(c) The following input-output pairs can be established:



Since we observe that $y[n] \neq y_1[n] + y_2[n]$ the system is not additive, and therefore not linear.

2. (5 marks) Let $x[n]$ an $v[n]$ be defined as follows:

$$x[n] = \delta[n-1] - \delta[n+1] \qquad \text{and} \qquad v[n] = a\delta[n] + b\delta[n-1] + c\delta[n-2].$$

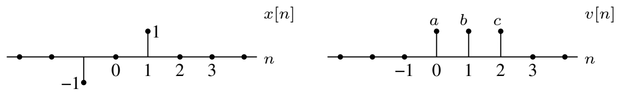
Suppose now that $y[n] = v[n] * x[n]$ gives

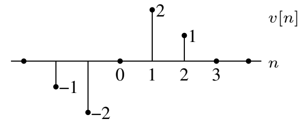
$$y[n] = -\delta[n+1] - 2\delta[n] + 2\delta[n-2] + \delta[n-3].$$

(a) Sketch the signals involved.

(b) Specify values for a, b, and c.

(a) Plots of the signals are as follows:





(b) It might be easier to solve via graphical convolution, but the answer can also be found algebraically:

$$\begin{aligned} y[n] = x[n] * v[n] &= (\delta[n-1] - \delta[n+1]) * (a\delta[n] + b\delta[n-1] + c\delta[n-2]) \\ &= a\delta[n-1] + b\delta[n-2] + c\delta[n-3] - a\delta[n+1] - b\delta[n] - c\delta[n-1] \\ &= -a\delta[n+1] - b\delta[n] + (a-c)\delta[n-1] + b\delta[n-2] + c\delta[n-3]. \end{aligned}$$

By inspection we observe that $a = 1$, $b = 2$, and $c = 1$.

3. (5 marks) Consider the discrete LTI system represented by

$$y[n] = x[n] - x[n-1]$$

where $x[n]$ and $y[n]$ are the input and output respectively.

(a) Determine and plot the impulse response $h[n]$. Is the system stable?

(b) Determine and plot the step response corresponding to $x[n] = u[n]$.

(c) Find $H(e^{j\omega})$ and plot its magnitude.

(d) Determine and plot the response to the input $x[n] = (-1)^n$.

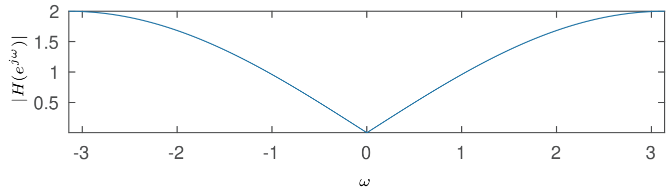
(a) The impulse response $h[n]$ is the output when the input is $x[n] = \delta[n]$, so $h[n] = \delta[n] - \delta[n-1]$. Since $\sum_{n=-\infty}^{\infty} |h[n]| = 2 < \infty$ the system is stable.

(b) When the input is $x[n] = u[n]$ then the output will be $y[n] = u[n] - u[n-1] = \delta[n]$.

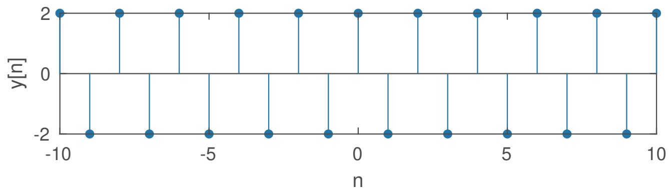
(c) The frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} (\delta[n] - \delta[n-1])\, e^{-j\omega n} = 1 - e^{-j\omega} = e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \\ &= 2je^{-j\omega/2}\sin(\omega/2), \end{aligned}$$

so $|H(e^{j\omega})| = 2|\sin(\omega/2)|$. Plot as follows:



(d) The response to $x[n] = (-1)^n = e^{j\pi n}$ will be $y[n] = H(e^{j\pi})e^{j\pi n} = 2e^{j\pi n} = 2(-1)^n$, as follows:



4. (5 marks) A LTI system has a step response of

$$g[n] = n\left(\frac{1}{2}\right)^n u[n].$$

In other words, when the input is $x[n] = u[n]$ then the output is $y[n] = g[n]$ above.

(a) Is the system causal? Why?

(b) Find the Fourier transform of $g[n]$.

(c) Find an expression for its impulse response. You may want to use the fact that $u[n] - u[n-1] = \delta[n]$.

(a) The system is causal because the impulse response is right-sided.

(b) By applying the frequency differentiation property to the pair

$$(1/2)^n u[n] \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (1 - 1/2e^{-j\omega})^{-1}$$

we find

$$\begin{aligned} G(e^{j\omega}) &= j\frac{d}{d\omega}(1 - 1/2e^{-j\omega})^{-1} = -j(1 - 1/2e^{-j\omega})^{-2} 1/2je^{j\omega} \\ &= \frac{1/2e^{j\omega}}{(1 - 1/2e^{-j\omega})^2}. \end{aligned}$$

(c) We are given that $u[n] \longrightarrow g[n]$ is a valid input-output pair. From time invariance $u[n-1] \longrightarrow g[n-1]$ is also a valid pair. By linearity

$$u[n] - u[n-1] \longrightarrow g[n] - g[n-1]$$

is valid, so

$$\delta[n] \longrightarrow g[n] - g[n-1] = n\left(\frac{1}{2}\right)^n u[n] - (n-1)\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

gives the impulse response.

Discrete-time Fourier transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$	Linearity
$x[n-n_d]$	$e^{-j\omega n_d}X(e^{j\omega})$	Time shift
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Common discrete-time Fourier transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega})=\begin{cases}1 & \lvert\omega\rvert < \omega_c\\ 0 & \omega_c < \lvert\omega\rvert \le \pi\end{cases}$
$x[n]=\begin{cases}1 & 0\le n\le M\\ 0 & \text{otherwise}\end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega-\omega_0+2\pi k)$

Z-transform properties

Sequences $x[n]$, $y[n]$	Transforms $X(z)$, $Y(z)$	ROC	Property
$ax[n]+by[n]$	$aX(z)+bY(z)$	ROC contains $R_x \cap R_y$	Linearity
$x[n-n_d]$	$z^{-n_d}X(z)$	ROC $= R_x$	Time shift
$z_0^n x[n]$	$X(z/z_0)$	ROC $= \lvert z_0\rvert R_x$	Frequency scale
$x^*[-n]$	$X^*(1/z^*)$	ROC $= \frac{1}{R_x}$	Time reversal
$nx[n]$	$-z\frac{dX(z)}{dz}$	ROC $= R_x$	Frequency diff.
$x[n]*y[n]$	$X(z)Y(z)$	ROC contains $R_x \cap R_y$	Convolution
$x^*[n]$	$X^*(z^*)$	ROC $= R_x$	Conjugation

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases}a^n & 0\le n\le N-1,\\ 0 & \text{otherwise}\end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$