EEE2047S EXAM
SIGNALS AND SYSTEMS I

University of Cape Town
Department of Electrical Engineering

November 2021
2 hours

Information

- The exam is closed-book.
- There are two parts to this exam. Each part must be answered and submitted in separate exam books.
- **Part A** has *four* questions totalling 40 marks. You must answer all of them.
- **Part B** has *three* questions totalling 30 marks. You must answer all of them.
- Marks are awarded based on method and clarity of presentation. Just writing down the answer is not a good strategy.
- The last page of this exam paper contains an information sheet with standard Fourier and Laplace transforms, transform properties, and some trigonometric identities.
- You have 2 hours.

PART A

The two parts of the exam must be answered in separate sets of exam books.

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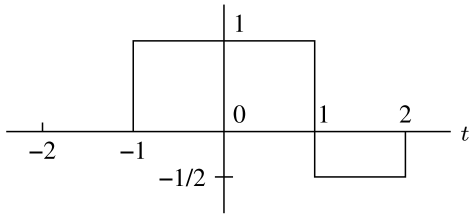
1. Let $g(t)$ be the function



Sketch the following:

(a) $y_1(t) = g(t-2)$

(b) $y_2(t) = g(2-t)$

(c) $y_3(t) = g(2t-2)$

(d) $y_4(t) = \int_{-\infty}^{t} g(\tau)d\tau$

(e) The generalised derivative $y_5(t) = \frac{d}{dt}g(t)$.

(10 marks)

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2. (a) Find the complex exponential Fourier series of the function

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

(b) The signal

$$g(t) = (1+j)e^{j\omega_0 t} + (1-j)e^{-j\omega_0 t}$$

can be written in the form

$$g(t) = a\cos(\omega_0 t + b)$$

for some values of a and b. Find these values.

(c) Find the Fourier transform of the signal

$$y(t) = \delta\left(\frac{t-b}{a}\right).$$

(10 marks)

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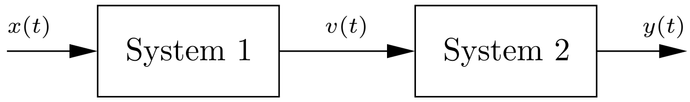
3. Consider the system below:



Suppose the two components are characterised by the following input-output relationships:

$$\text{System 1: } v(t) = h_1(t) * x(t) \quad \text{with} \quad h_1(t) = u(t)$$

$$\text{System 2: } y(t) = \int_{t-3}^{t} v(\tau) d\tau.$$

It is easy to show that System 2 is linear, and you may assume this to be true without proving it.

(a) Show that System 2 is time invariant.

(b) Is the overall system linear and time invariant? Why?

(c) Show that the impulse response of System 2 is $h(t) = u(t) - u(t-3)$.

(d) Determine and plot the impulse response of the overall system.

(10 marks)

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4. The following is a valid Fourier transform pair:

$$e^{-|t|} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{2}{\omega^2 + 1}.$$

(a) Sketch the signal $x_a(t) = 4e^{-|t|/16}$.

(b) Find the Fourier transform of $x_a(t)$, and sketch the corresponding magnitude and phase.

(c) Find the Fourier transform of $x_b(t) = \frac{1}{(t-1)^2+1}$.

(10 marks)

PART B

The two parts of the exam must be answered in separate sets of exam books.

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5. A system is defined by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 12y(t) = \frac{dx(t)}{dt} - 3x(t).$$

   (a) What is the transfer function $H(s)$ of this system? Comment on the stability of the system.

   (b) If $y(0^-) = 1$ and $\dot{y}(0^-) = 3$, what is the zero input response of this system? Comment on the properties of this response.

(10 marks)

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6. A system has a transfer function given by

$$H(s) = \frac{1}{s^2 + 6s + 10}.$$

   (a) Show that its impulse response is $h(t) = e^{-3t}\sin(t)u(t)$.

   (b) What is the output when the input $x(t) = e^{-2t}u(t)$?

(10 marks)

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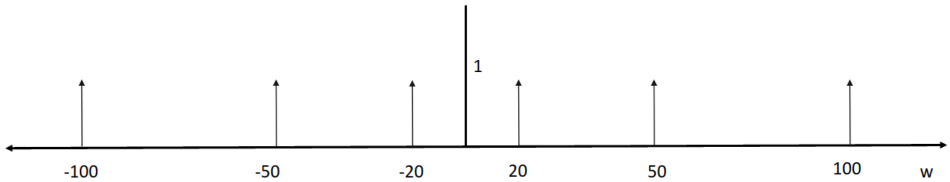
7. The frequency response of a filter is defined as follows:

$$H(\omega) = \frac{2500}{2500 + 70j\omega - \omega^2}.$$

(a) Sketch the magnitude of the frequency response for the range $-1000 \leq \omega \leq 1000$.

(b) What type of filter is this?

(c) Consider the following signal $X(\omega)$ being passed through this filter:



Calculate the output $y(t)$.

(10 marks)

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc}\frac{\tau\omega}{2\pi}$
$\tau \text{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t+\tau/2) - u(t-\tau/2)$ and $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(s) * V(s)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$
Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $b = \sqrt{c-a^2}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$