

EEE2047S EXAM SIGNALS AND SYSTEMS I

University of Cape Town
Department of Electrical Engineering

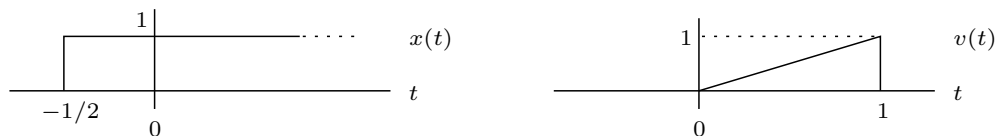
November 2018

2 hours

Information

- The exam is closed-book.
 - There are 8 questions totaling 75 marks. You must answer all of them.
 - Marks are awarded based on method and clarity of presentation. Just writing down the answer is not a good strategy.
 - The last two pages of this exam paper contain an information sheet with standard Fourier and Laplace transforms, transform properties, and some trigonometric identities.
 - You have 2 hours.
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1. Compute the convolution of the following two signals:



(10 marks)

2. Consider the system



with input-output relationship

$$y(t) = \int_{t-1/2}^{t+1/2} x(\tau) d\tau.$$

- Is the system linear? Why?
- Is the system time invariant? Why?
- Is the system causal? Why?
- Show that the impulse response of the system is $h(t) = p_1(t)$.
- Find and sketch the output of the system when the input is the unit step $x(t) = u(t)$.

(10 marks)

3. Suppose the signal $x(t)$ can be written as an exponential Fourier series

$$x(t) = e^{-2jt} + je^{-jt} + 2 - je^{jt} + e^{2jt}.$$

- What is the fundamental period of $x(t)$?
- Plot the magnitude and phase of the Fourier series coefficients of $x(t)$.
- Since $x(t)$ is real it can be written in trigonometric form

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t + b_k).$$

Specify the value of a_0 , and the values of a_k and b_k for all $k \geq 1$.

(10 marks)

4. Find the impulse response of the system with frequency response

$$H(\omega) = \frac{j\omega}{j\omega + 4} - \frac{1}{(j\omega + 4)^2}.$$

For the second term it is useful to apply the frequency differentiation property to the pair

$$e^{-bt}u(t) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{j\omega + b} \quad (b > 0).$$

(10 marks)

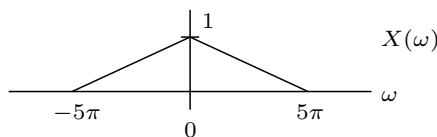
5. A linear system has frequency response

$$H(\omega) = \frac{j\omega}{1 + j\omega}.$$

- What is the DC gain of the system?
- Roughly sketch $|H(\omega)|$. What kind of system does this represent?
- Find the impulse response of the system.
- Determine the system response to $x(t) = \sin(2t)$.

(10 marks)

6. Suppose $x(t)$ is a signal with the spectrum shown below:



- Find the energy contained in $x(t)$.
- Suppose $y_1(t) = x(t) \cos(15\pi t)$. Plot $Y_1(\omega)$.
- Suppose $y_2(t) = p(t)x(t)$ with

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

and $T = 2/5$. Plot $Y_2(\omega)$.

(10 marks)

7. A causal system has a Laplace transform

$$F(s) = \frac{s^2 - 4}{s^2 + 6s + 9}.$$

- (a) Specify the locations of the poles and zeros of the system, and draw a pole-zero plot. Is the system stable? Why?
- (b) Find a time-domain expression for the response of the system to a unit step $x(t) = u(t)$.

(10 marks)

8. Consider the initial value problem

$$\frac{d^2}{dt^2}y(t) + 7\frac{d}{dt}y(t) + 12y(t) = 0$$

subject to $y(0) = 1$ and $\frac{d}{dt}y(t) = 2$. Determine $Y(s)$.

(5 marks)

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \iff aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \iff X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \iff \frac{1}{ a }X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \iff X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \iff j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Frequency shift	$x(t)e^{j\omega_0 t} \iff X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t) \cos(\omega_0 t) \iff \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \iff (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \iff \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \iff X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \iff \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) ^2 d\omega$
Duality	$X(t) \iff 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$
1 $(-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad (c \text{ any real number})$
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad (\omega_0 \text{ any real number})$
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2 t }{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \text{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2 \omega }{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$ and $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \iff aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \iff e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \iff \frac{1}{a}X\left(\frac{s}{a}\right) \quad a > 0$
Frequency differentiation	$t^n x(t) \iff (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \iff X(s-a)$
Differentiation	$x'(t) \iff sX(s) - x(0^-)$ $x''(t) \iff s^2X(s) - sx(0^-) - x'(0^-)$ $x^{(n)}(t) \iff s^n X(s) - s^{n-1}x(0^-) - \dots - x^{(n-1)}(0^-)$
Integration	$\int_0^t x(\lambda)d\lambda \iff \frac{1}{s}X(s)$ $\int_{-\infty}^t x(\lambda)d\lambda \iff \frac{1}{s}X(s) + \frac{1}{s} \int_{-\infty}^0 x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \iff X(s)V(s)$
Frequency convolution	$x(t)v(t) \iff \frac{1}{2\pi j} X(s) * V(s)$

Initial value: $f(0^+) = \lim_{s \rightarrow \infty} sF(s)$

Final value: $f(\infty) = \lim_{s \rightarrow 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st} ds$	$X(s) = \int_0^\infty x(t)e^{-st} dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t} u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t} u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t} u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at} \cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at} \sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{(r \cos \theta)s + (ar \cos \theta - br \sin \theta)}{s^2 + 2as + (a^2 + b^2)}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}, b = \sqrt{c-a^2}, \theta = \tan^{-1} \frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$$\sin(-\theta) = -\sin(\theta) \quad \cos(-\theta) = \cos(\theta) \quad \tan(-\theta) = -\tan(\theta) \quad \sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta) \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2 \cos^2(\theta) - 1 = 1 - 2 \sin^2(\theta)$$

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2) \quad \cos(\theta_1 + \theta_2) = \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)$$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$