EEE2047S EXAM
SIGNALS AND SYSTEMS I

University of Cape Town
Department of Electrical Engineering

November 2017
2 hours

Information

- The exam is closed-book.
- There are *7* questions totaling 70 marks. You must answer all of them.
- Marks are awarded based on method and clarity of presentation. Just writing down the answer is not a good strategy.
- The last page of this exam paper contains an information sheet with standard Fourier transforms, transform properties, and some trigonometric identities.
- You have 2 hours.

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1. A moving average LTI system has impulse response

$$h(t) = p_1(t - 1/2),$$

where $p_1(t)$ is the pulse of unit width and unit height centered on the origin.

(a) Is the system causal? Why?

(b) Find and sketch the output under initial rest conditions when the input is $x(t) = 2e^{-3t}u(t)$.

(c) Find the frequency response of the system.

(10 marks)

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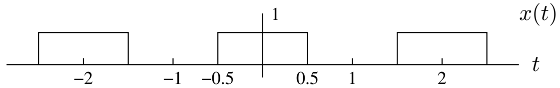
2. The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi}\sin(k\pi/2) & \text{otherwise.} \end{cases}$$

(a) How much signal power is contained in the first harmonic?

(b) Show that the signal $y(t) = x(t)e^{j5\pi t}$ can be written in the form

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$$

with $d_k = c_{k-5}$.

(c) Plot the magnitude and phase of these coefficients d_k for $k = 1$ to $k = 9$.

(10 marks)

3. Recall Euler's formula $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Answer the following two questions, which do not depend on one another:

(a) The signal $f(t) = -2\cos(\omega_0 t) + 2\sqrt{3}\sin(\omega_0 t + \pi/3)$ can be written in the form

$$f(t) = ce^{j\omega_0 t} + c^* e^{-j\omega_0 t},$$

where c^* is the conjugate of c. Find the value of c in this representation.

(b) The signal

$$g(t) = (1+j)e^{j\omega_0 t} + (1-j)e^{-j\omega_0 t}$$

can be written in the form

$$g(t) = a\cos(\omega_0 t + b)$$

for some values of a and b. Find these values.

(10 marks)

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4. (a) Simplify the following expression: $w(t) = \delta(t-3)\frac{4-jt^2}{2t}$.

(b) Find the Fourier transform of $x(t) = (\cos(5t) + e^{-2t})u(t)$.

(c) Find the inverse Fourier transform of $Y(\omega) = \frac{5\omega}{j\omega+1}$.

(d) Suppose $\pi_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ and $x(t) = \text{sinc}\left(\frac{t}{2\pi}\right)$. Sketch $Y(\omega)$, the Fourier transform of $y(t) = \pi_T(t) * x(t)$ for $T = 2$.

(10 marks)

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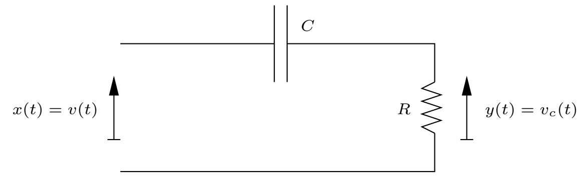
5. The RC highpass circuit below



has an impulse response given by

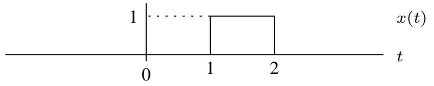
$$h(t) = \delta(t) - \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t),$$

where $\tau = RC$ is the time constant of the network.

(a) Write down a differential equation relating the input and the output of the system.

(b) Show that the step response of the network, namely the output when the input is $x(t) = u(t)$, is given by $g(t) = e^{-\frac{t}{\tau}}u(t)$.

(c) Using this result, find the response of the circuit to the input below, under an initial rest assumption:



(10 marks)

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6. (a) Consider the right-sided function

$$g(t) = \begin{cases} t & 0 \le t < 2 \\ 4 & t \ge 2. \end{cases}$$

Show that this can be written as

$$g(t) = tu(t) - (t-2)u(t-2) + 2u(t-2)$$

and find its Laplace transform.

(b) Find the inverse Laplace transform of $F(s) = \frac{s+3}{s^2+4s+29}$.

(10 marks)

7. A causal system obeys the following differential equation:

$$\frac{d^2}{dt^2}w(t) + w(t) = x(t),$$

where $x(t)$ is the input and $w(t)$ is the output.

(a) Find the location of the poles and the zeros of the system. Is it stable? Why?

(b) Use the Laplace transform to find the output signal $w(t)$ when the input is $x(t) = (t^2 + 2)u(t)$. Assume the initial conditions $w(0^-) = 1$ and $w'(0^-) = -1$, where $w'(t)$ is the first derivative of $w(t)$ with respect to t.

(10 marks)

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc}\frac{\tau\omega}{2\pi}$
$\tau \text{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right)p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right)p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t+\tau/2) - u(t-\tau/2)$ and $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(s) * V(s)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$

Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $b = \sqrt{c-a^2}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$