

# EEE2047S EXAM SIGNALS AND SYSTEMS I

University of Cape Town  
Department of Electrical Engineering

November 2017

2 hours

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## Information

- The exam is closed-book.
  - There are 7 questions totaling 70 marks. You must answer all of them.
  - Marks are awarded based on method and clarity of presentation. Just writing down the answer is not a good strategy.
  - The last page of this exam paper contains an information sheet with standard Fourier transforms, transform properties, and some trigonometric identities.
  - You have 2 hours.
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1. A moving average LTI system has impulse response

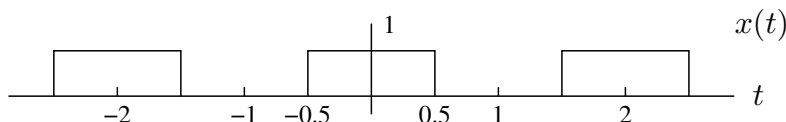
$$h(t) = p_1(t - 1/2),$$

where  $p_1(t)$  is the pulse of unit width and unit height centered on the origin.

- Is the system causal? Why?
- Find and sketch the output under initial rest conditions when the input is  $x(t) = 2e^{-3t}u(t)$ .
- Find the frequency response of the system.

(10 marks)

2. The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi} \sin(k\pi/2) & \text{otherwise.} \end{cases}$$

- How much signal power is contained in the first harmonic?
- Show that the signal  $y(t) = x(t)e^{j5\pi t}$  can be written in the form

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$$

with  $d_k = c_{k-5}$ .

- Plot the magnitude and phase of these coefficients  $d_k$  for  $k = 1$  to  $k = 9$ .

(10 marks)

3. Recall Euler's formula  $e^{j\theta} = \cos(\theta) + j \sin(\theta)$ . Answer the following two questions, which do not depend on one another:

(a) The signal  $f(t) = -2 \cos(\omega_0 t) + 2\sqrt{3} \sin(\omega_0 t + \pi/3)$  can be written in the form

$$f(t) = ce^{j\omega_0 t} + c^* e^{-j\omega_0 t},$$

where  $c^*$  is the conjugate of  $c$ . Find the value of  $c$  in this representation.

(b) The signal

$$g(t) = (1 + j)e^{j\omega_0 t} + (1 - j)e^{-j\omega_0 t}$$

can be written in the form

$$g(t) = a \cos(\omega_0 t + b)$$

for some values of  $a$  and  $b$ . Find these values.

(10 marks)

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4. (a) Simplify the following expression:  $w(t) = \delta(t - 3) \frac{4 - jt^2}{2t}$ .

(b) Find the Fourier transform of  $x(t) = (\cos(5t) + e^{-2t})u(t)$ .

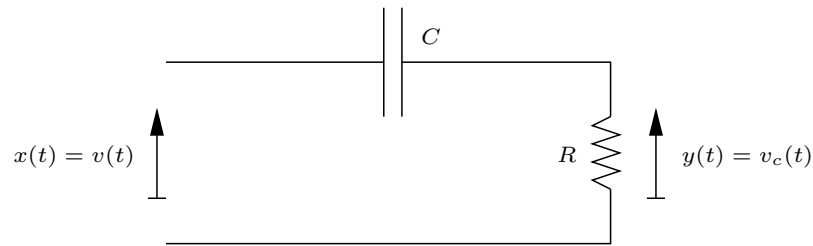
(c) Find the inverse Fourier transform of  $Y(\omega) = \frac{5\omega}{j\omega + 1}$ .

(d) Suppose  $\pi_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$  and  $x(t) = \text{sinc}\left(\frac{t}{2\pi}\right)$ . Sketch  $Y(\omega)$ , the Fourier transform of  $y(t) = \pi_T(t) * x(t)$  for  $T = 2$ .

(10 marks)

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5. The RC highpass circuit below

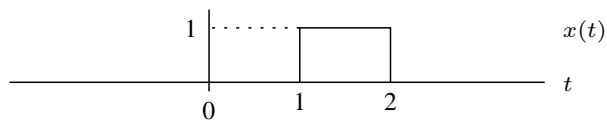


has an impulse response given by

$$h(t) = \delta(t) - \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t),$$

where  $\tau = RC$  is the time constant of the network.

- Write down a differential equation relating the input and the output of the system.
- Show that the step response of the network, namely the output when the input is  $x(t) = u(t)$ , is given by  $g(t) = e^{-\frac{t}{\tau}} u(t)$ .
- Using this result, find the response of the circuit to the input below, under an initial rest assumption:



(10 marks)

6. (a) Consider the right-sided function

$$g(t) = \begin{cases} t & 0 \leq t < 2 \\ 4 & t \geq 2. \end{cases}$$

Show that this can be written as

$$g(t) = tu(t) - (t-2)u(t-2) + 2u(t-2)$$

and find its Laplace transform.

- Find the inverse Laplace transform of  $F(s) = \frac{s+3}{s^2+4s+29}$ .

(10 marks)

7. A causal system obeys the following differential equation:

$$\frac{d^2}{dt^2}w(t) + w(t) = x(t),$$

where  $x(t)$  is the input and  $w(t)$  is the output.

- (a) Find the location of the poles and the zeros of the system. Is it stable? Why?
- (b) Use the Laplace transform to find the output signal  $w(t)$  when the input is  $x(t) = (t^2 + 2)u(t)$ . Assume the initial conditions  $w(0^-) = 1$  and  $w'(0^-) = -1$ , where  $w'(t)$  is the first derivative of  $w(t)$  with respect to  $t$ .

(10 marks)

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# INFORMATION SHEET

## Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \iff aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \iff X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \iff \frac{1}{ a }X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \iff X(-\omega) = \overline{X(\omega)}$
Multiplication by power of $t$	$t^n x(t) \iff j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Frequency shift	$x(t)e^{j\omega_0 t} \iff X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t) \cos(\omega_0 t) \iff \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \iff (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \iff \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \iff X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \iff \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^2 d\omega$
Duality	$X(t) \iff 2\pi x(-\omega)$

## Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$
1 $(-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad (c \text{ any real number})$
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad (\omega_0 \text{ any real number})$
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2 t }{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \text{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2 \omega }{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

with  $p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$  and  $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$ .

# Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \iff aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \iff e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \iff \frac{1}{a}X\left(\frac{s}{a}\right) \quad a > 0$
Frequency differentiation	$t^n x(t) \iff (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \iff X(s-a)$
Differentiation	$x'(t) \iff sX(s) - x(0^-)$ $x''(t) \iff s^2X(s) - sx(0^-) - x'(0^-)$ $x^{(n)}(t) \iff s^n X(s) - s^{n-1}x(0^-) - \dots - x^{(n-1)}(0^-)$
Integration	$\int_0^t x(\lambda)d\lambda \iff \frac{1}{s}X(s)$ $\int_{-\infty}^t x(\lambda)d\lambda \iff \frac{1}{s}X(s) + \frac{1}{s} \int_{-\infty}^0 x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \iff X(s)V(s)$
Frequency convolution	$x(t)v(t) \iff \frac{1}{2\pi j} X(s) * V(s)$

Initial value:  $f(0^+) = \lim_{s \rightarrow \infty} sF(s)$

Final value:  $f(\infty) = \lim_{s \rightarrow 0} sF(s)$  with all poles in left-hand plane

## Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st} ds$	$X(s) = \int_0^{\infty} x(t)e^{-st} dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t} u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t} u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t} u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at} \cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at} \sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{(r \cos \theta)s + (ar \cos \theta - br \sin \theta)}{s^2 + 2as + (a^2 + b^2)}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}, b = \sqrt{c-a^2}, \theta = \tan^{-1} \frac{Aa-b}{A\sqrt{c-a^2}}$

## Trigonometric identities

$$\sin(-\theta) = -\sin(\theta) \quad \cos(-\theta) = \cos(\theta) \quad \tan(-\theta) = -\tan(\theta) \quad \sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta) \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2 \cos^2(\theta) - 1 = 1 - 2 \sin^2(\theta)$$

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2) \quad \cos(\theta_1 + \theta_2) = \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)$$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$