EEE2035F EXAM
SIGNALS AND SYSTEMS I

HINTS: June 2016

University of Cape Town
Department of Electrical Engineering

June 2016
2 hours

Information

- The exam is closed-book.
- There are *8* questions totaling 75 marks. You must answer all of them.
- Marks are awarded based on method and clarity of presentation. Just writing down the answer is not a good strategy.
- The last page of this exam paper contains an information sheet with standard Fourier transforms, transform properties, and some trigonometric identities.
- You have 2 hours.

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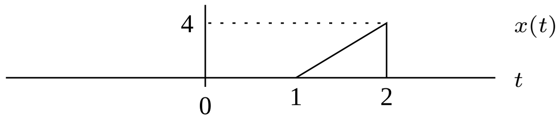
1. Consider the signal below:



Plot the following:

(a) $y_1(t) = x(1-t)$

(b) $y_2(t) = x(t+1/2)\delta(t-1)$

(c) $y_3(t) = x(t/2-1)$

(d) $y_4(t) = \int_{-\infty}^{t} x(\lambda)d\lambda$

(e) $y_5(t) = \frac{d}{dt}x(t)$.

(10 marks)

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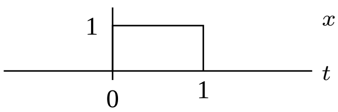
2. Note that the systems in the three parts of this question are different.

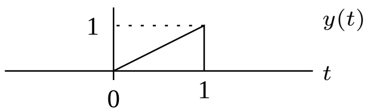
(a) The input $x(t)$ and the output $y(t)$ of a system T_a are related by the equation

$$y(t) = x(t-1) + x(1-t).$$

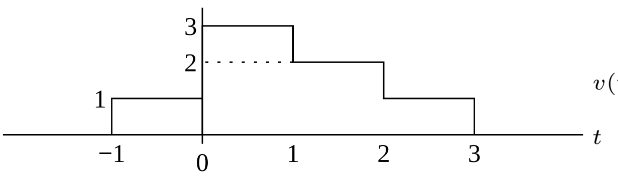
Is the system time invariant?

(b) When the input to a LTI system T_b is $x(t)$ below then the output is $y(t)$:





Find the output when the input is the following:



(c) A LTI system T_c with input $x(t)$ and output $y(t)$ is governed by the input-output relationship

$$y(t) = \frac{1}{3}(x(t+1) + x(t) + x(t-1)).$$

Plot the impulse response of the system.

(10 marks)

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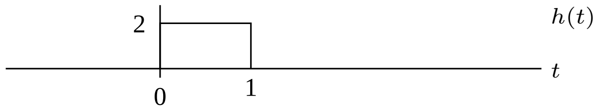
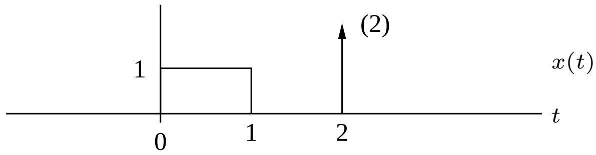
3. A system with impulse response



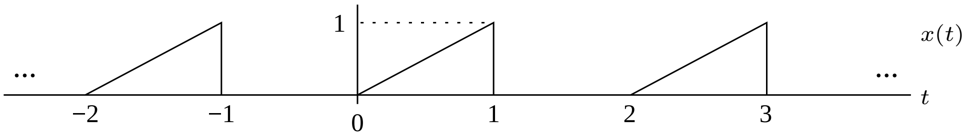
is driven by an input signal



Find and plot the output $y(t)$.

(10 marks)

4. The signal



can be written in the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

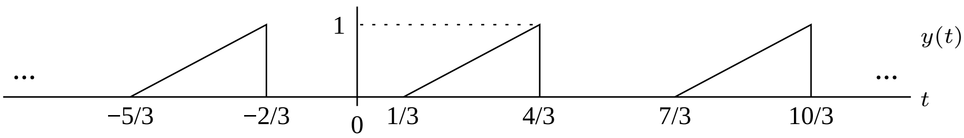
where the coefficients satisfy

$$c_k = \begin{cases} 1/4 & (k = 0) \\ \frac{1}{jk\pi}\left[\frac{1}{k\pi}\sin(k\pi/2)e^{-jk\pi/2} - \frac{1}{2}e^{-jk\pi}\right] & (\text{otherwise}). \end{cases}$$

(a) Show that $|c_2| = \frac{1}{4\pi}$ and $\angle c_2 = \frac{\pi}{2}$. What frequency is associated with this coefficient?

(b) What is the value of the coefficient c_{-2}?

(c) The signal



can be written in the form $y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$. Show that $d_k = c_k e^{-jk\pi/3}$.

(d) What are the values $|d_2|$ and $\angle d_2$?

(10 marks)

5. A LTI system generates the output

$$y(t) = (e^{-2t} - e^{-3t})u(t)$$

in response to the input $x(t) = e^{-2t}u(t)$.

(a) Show that the frequency response of the system is

$$H(\omega) = \frac{1}{j\omega + 3}.$$

(b) Determine the unit impulse response $h(t)$ of the system.

(c) Sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$ of the system.

(d) What is the output $y(t)$ of the system when the input is $x(t) = e^{j3t}$?

(10 marks)

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6. A system with input $x(t)$ and output $y(t)$ is described by the differential equation

$$\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = -\frac{d}{dt}x(t).$$

(a) Show that the frequency response of the system is

$$H(\omega) = \frac{-j\omega}{(j\omega - 3)(j\omega - 2)}.$$

(b) Find the impulse response of the system. You can use the partial fraction expansion below if necessary:

$$\frac{-1}{(s-2)(s-3)} = \frac{1}{(s-2)} - \frac{1}{(s-3)}.$$

(10 marks)

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7. The Fourier transform of signal $x(t)$ is given by

$$X(\omega) = \frac{2}{(j\omega)}(1 - \cos(\omega)).$$

(a) Without calculating $x(t)$ find the Fourier transform $Y(\omega)$ of $y(t) = x(0.5t - 2)$.

(b) Show that the following is a valid Fourier pair:

$$\frac{1}{2}(\delta(t-1) + \delta(t+1)) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \cos(\omega).$$

(c) Find the inverse Fourier transform $x(t)$. You may use the result from (b) if required.

(10 marks)

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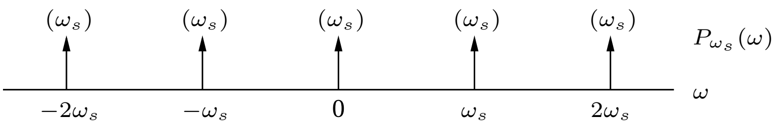
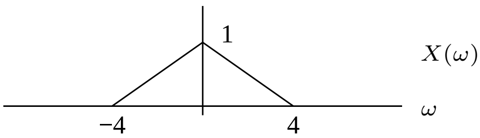
8. The periodic impulse train signal

$$P_{\omega_s}(\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

is sketched below:



(a) Suppose the signal $x(t)$ has the following spectrum:



Sketch $Y(\omega) = P_{\omega_s}(\omega) * X(\omega)$ for $\omega_s = 10$.

(b) What is the smallest value of ω_s for which the signal $x(t)$ can be recovered from $Y(\omega) = P_{\omega_s}(\omega) * X(\omega)$?

(c) Specify a filter $H(\omega)$ that can be used to reconstruct $x(t)$ from $Y(\omega)$ in part (a).

(5 marks)

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau\mathrm{sinc}\frac{\tau\omega}{2\pi}$
$\tau\mathrm{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\mathrm{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$ and $\mathrm{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(\omega) * V(\omega)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$
Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$re^{-\alpha t}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-\alpha t}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-\alpha t}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$, $b = \sqrt{c-a^2}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$