# EEE2035F EXAM SIGNALS AND SYSTEMS I

University of Cape Town Department of Electrical Engineering

> June 2015 2 hours

#### Information

- The exam is closed-book.
- There are  $\delta$  questions totaling 70 marks. You must answer all of them.
- The last page of this exam paper contains an information sheet with standard Fourier transforms, transform properties, and some trigonometric identities.
- You have 2 hours.

1. Let q(t) be the function



Sketch the following:

- (a)  $y_1(t) = g(t-2)$
- (b)  $y_2(t) = g(2-t)$
- (c)  $y_3(t) = -2g(2t)$
- (d)  $y_4(t) = g(2t 2)$
- (e)  $y_5(t) = \int_{-\infty}^t g(\tau) d\tau$
- (f) The generalised derivative  $y_6(t) = \frac{d}{dt}g(t)$ .

(10 marks)

2. The charge q(t) accumulated by an ideal capacitor C is related to the current i(t) passing through it by

$$q(t) = \int_{-\infty}^{t} i(\phi) d\phi$$

- (a) With the input i(t) and output q(t), what is the impulse response of the capacitor?
- (b) Is the system causal? Does it have memory? Explain, with reasons.
- (c) Show that the system is time invariant.

(5 marks)

3. Find and plot y(t) = h(t) \* x(t) with

 $h(t) = e^{-t}u(t)$  and x(t) = u(t-1).

(5 marks)

4. A linear time-invariant system has the following step response

$$g(t) = 2e^{-4t}u(t+1).$$

- (a) Sketch the step response as function of time.
- (b) Find the impulse response h(t).
- (c) Is this system causal or non-causal? Justify your answer.
- (d) What is the output if the input is x(t) = 3u(t+4)?

5. The signal



(10 marks)

has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0\\ \frac{1}{k\pi} \sin(k\pi/2) & \text{otherwise.} \end{cases}$$

- (a) Plot the magnitude and phase of the coefficients  $c_k$  as a function of k over the range -4 to 4.
- (b) How much power is contained in the third harmonic of x(t)?
- (c) Use the information given to find a Fourier series expansion for the signal y(t) below:



6. The input x(t) and output y(t) of a particular RC circuit obeys the differential equation

$$y(t) = RC\left(\frac{dx(t)}{dt} - \frac{dy(t)}{dt}\right).$$

(a) Show that the frequency response of the system is

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC},$$

- (b) Plot  $|H(\omega)|$ . Does the circuit have a lowpass or a highpass transfer characteristic?
- (c) Find the impulse response of the system.
- (d) What is the output of the system if the input is  $x(t) = \cos(5\pi t)$  and RC = 1? This output should be real but you may express your result in terms of complex quantities.

(10 marks)

7. (a) Find the inverse Fourier transform of the function

$$X(\omega) = \frac{6}{3+j\omega-j4}e^{j\omega}.$$

(b) Use the valid transform pair

$$e^{-|t|} \xrightarrow{\mathcal{F}} \frac{2}{1+\omega^2}$$

to find the Fourier transform of

$$x(t) = \frac{e^{jWt}}{a^2 + t^2}.$$

(10 marks)

8. Consider a signal g(t) with the following spectrum:



- (a) Plot the frequency spectrum of  $f(t) = g(t) \cos(20t)$ .
- (b) Suppose f(t) is now filtered using an LTI system with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \le 20\\ 0 & \text{otherwise.} \end{cases}$$

to produce the output q(t). Plot the frequency spectrum  $Q(\omega)$  of q(t).

- (c) Calculate the total energy in q(t).
- (d) According to Nyquist, what is the largest sampling interval T for which the signal g(t) can be reconstructed from samples g[n] = g(nT)?
- (e) Sketch the spectrum of  $g_s(t) = g(t)p(t)$ , where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - T_s k)$$

and  $T_s = \pi/15$  seconds.

(10 marks)

### **INFORMATION SHEET**

#### Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omegac}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a})  a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of $t$	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega)  n = 1, 2, \dots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)  \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t)\leftrightarrow (j\omega)^n X(\omega)  n=1,2,\ldots$
Integration	$\int_{-\infty}^{t} x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^{2}(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^{2} d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

## **Common Fourier Transform Pairs**

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
$1  (-\infty < t < \infty)$	$2\pi\delta(\omega)$
-0.5 + u(t)	$\frac{1}{j\omega}$
u(t)	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c}$ ( <i>c</i> any real number)
$e^{-bt}u(t)$	$\frac{1}{i\omega+b}$ $(b>0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$ ( $\omega_0$ any real number)
$p_{ au}(t)$	$ au \operatorname{sinc} \frac{ au \omega}{2\pi}$
$ au \operatorname{sinc} \frac{ au t}{2\pi}$	$2\pi p_{ au}(\omega)$
$\left(1-rac{2 t }{ au} ight)p_{ au}(t)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\operatorname{sinc}^2\frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2 \omega }{\tau}\right) p_{\tau}(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0)+e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi [e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{T})$

## Trigonometric identities

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\begin{aligned} \sin(-\theta) &= -\sin(\theta) & \cos(-\theta) = \cos(\theta) & \tan(-\theta) = -\tan(\theta) & \sin^2(\theta) + \cos^2(\theta) = 1\\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta) & \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)\\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\\ e^{j\theta} &= \cos(\theta) + j\sin(\theta) \end{aligned}
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