EEE235F MOCK EXAM
SIGNALS AND SYSTEMS I

University of Cape Town
Department of Electrical Engineering

June 2005
2 hours

Information

- The exam is closed-book.
- There are *eight* questions totalling 80 marks. You must answer all of them.
- The last page of this exam paper contains an information sheet with standard Fourier transforms and transform properties.
- You have 2 hours.

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1. The input $x(t)$ and the impulse response $h(t)$ of a continuous time LTI system are given by

$$x(t) = u(t) \qquad \text{and} \qquad h(t) = e^{-\alpha t}u(t), \quad \alpha > 0.$$

   (a) Compute the output $y(t)$ using $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$.

   (b) Compute the output $y(t)$ using $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$.

(10 marks)

(a) The first integral is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)e^{-\alpha(t-\tau)}u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-\alpha(t-\tau)}u(t-\tau)d\tau \\ &= \begin{cases} \int_0^t e^{-\alpha(t-\tau)}d\tau & (t \geq 0) \\ 0 & (t < 0) \end{cases} \\ &= u(t)\int_0^t e^{-\alpha t}e^{\alpha\tau}d\tau = u(t)e^{-\alpha t}\int_0^t e^{\alpha\tau}d\tau = u(t)e^{-\alpha t}\left[\frac{1}{\alpha}e^{\alpha\tau}\right]_{\tau=0}^{\tau=t} \\ &= u(t)\frac{1}{\alpha}e^{-\alpha t}(e^{\alpha t}-1) = \frac{1}{\alpha}(1-e^{-\alpha t})u(t). \end{aligned}$$

(b) The second integral is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-\alpha\tau}u(t-\tau)d\tau \\ &= \begin{cases} \int_0^t e^{-\alpha\tau}d\tau & (t \geq 0) \\ 0 & (t < 0) \end{cases} \\ &= u(t)\int_0^t e^{-\alpha\tau}d\tau = u(t)\left[-\frac{1}{\alpha}e^{-\alpha\tau}\right]_{\tau=0}^{\tau=t} = \frac{1}{\alpha}(1-e^{-\alpha t})u(t). \end{aligned}$$

Naturally the two are equal, as required.

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2. Let $x(t)$ and $v(t)$ be the signals

$$x(t) = u(t) \qquad \text{and} \qquad v(t) = e^{-\alpha t}u(t), \quad \alpha > 0.$$

Sketch the following:

(a) $x(t-\lambda)v(\lambda)$ for $t=1$, $t=\frac{1}{2}$, and $t=0$.

(b) Hence find $w(0)$, $w(\frac{1}{2})$ and $w(1)$ if $w(t)=x(t)*v(t)$.

(c) Also sketch $x(t-\lambda)\delta(\lambda)$ for $t=-\frac{1}{2}$ and $t=\frac{1}{2}$.

(10 marks)

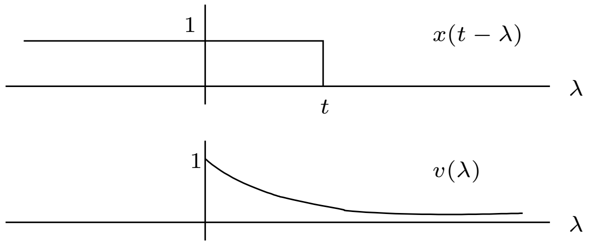
(a) For t fixed, $x(t-\lambda)v(\lambda)$ is a function of λ. The components of this product are

$$x(t-\lambda)=u(t-\lambda)$$

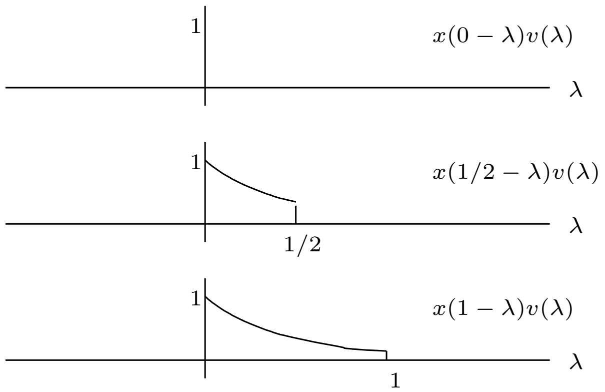
and

$$v(\lambda)=e^{-\alpha\lambda}u(\lambda),$$

which when plotted look as follows:



A change in t just causes a shift in the position of the step in $x(t-\lambda)$. Thus



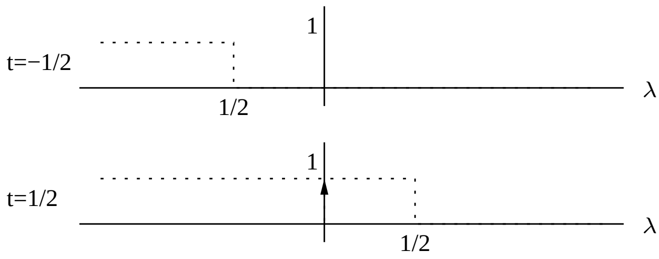
(b) The convolution output for the three points specified is the area under the three curves given above. Thus $w(0)=0$, and

$$w(1/2)=\int_0^{1/2} e^{-\alpha t}dt=\frac{1}{\alpha}(e^{-\alpha/2}-1)$$

and

$$w(1)=\int_0^{1} e^{-\alpha t}dt=\frac{1}{\alpha}(e^{-\alpha}-1).$$

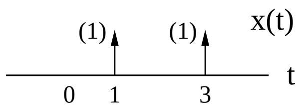
(c) The function $x(t-\lambda)\delta(\lambda)$ is sketched below for the three different values of $t=-1/2$ and $t=1/2$:



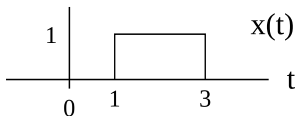
3. Consider a LTI system with *step* response

$$g(t) = e^{-t}u(t).$$

(a) Determine and sketch the response to



(b) Determine and sketch the response to



(10 marks)

(a) If $g(t)$ is the step response of the system, then it is the output of the system when the input is the unit step

$$x(t) = u(t) \longrightarrow y(t) = g(t).$$

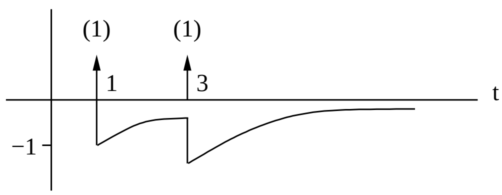
Taking derivatives with respect to time shows that the impulse response of the system is the derivative $g'(t)$ of the impulse response. Thus the impulse response is

$$h(t) = g'(t) = \delta(t) - e^{-t}u(t).$$

Through time shift and linearity the response to the signal $x(t) = \delta(t-1) + \delta(t-3)$ is

$$\begin{aligned} y(t) &= h(t-1) + h(t-3) \\ &= \delta(t-1) - e^{-(t-1)}u(t-1) + \delta(t-3) - e^{-(t-3)}u(t-3), \end{aligned}$$

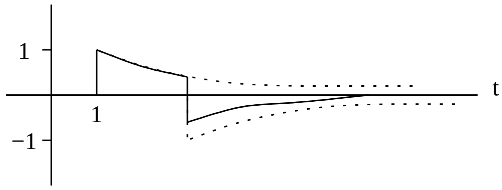
which looks like this:



(b) The second input can be expressed as $x(t) = u(t-1) - u(t-3)$, so by linearity and time invariance the corresponding output is

$$y(t) = g(t-1) - g(t-3) = e^{-(t-1)}u(t-1) - e^{-(t-3)}u(t-3)$$

A plot of this looks something like this:



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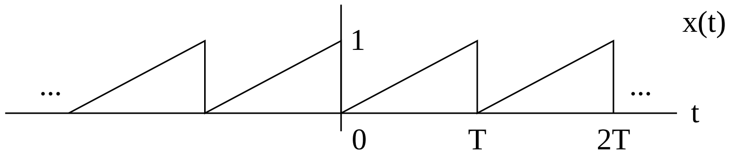
4. Find the trigonometric form with real coefficients of the Fourier series for the following signal:



(10 marks)

First find the Fourier series in terms of complex exponentials: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$, with $\omega_0 = 2\pi/T$, $c_0 = 1/2$, and

$$\begin{aligned}
c_k &= \frac{1}{T}\int_0^T t/T e^{-j\omega_0 kt} dt = \frac{1}{T^2}\left\{\left[\frac{t}{-j\omega_0 k} e^{-j\omega_0 kt}\right]_{t=0}^{t=T} - \int_0^T \frac{1}{-j\omega_0 k} e^{-j\omega_0 kt} dt\right\} \\
&= \frac{1}{T^2}\left\{\frac{1}{-j\omega_0 k}\left[te^{-j\frac{2\pi}{T}kt}\right]_{t=0}^{t=T} - \frac{1}{(j\omega_0 k)^2}\left[e^{-j\frac{2\pi}{T}kt}\right]_{t=0}^{t=T}\right\} \\
&= \frac{1}{T^2}\left\{\frac{1}{-j\omega_0 k}(T-0) + \frac{1}{(\omega_0 k)^2}(1-1)\right\} = -\frac{1}{j\frac{2\pi}{T}kT} = \frac{j}{2\pi k}.
\end{aligned}$$

The Fourier series for $x(t)$ can therefore be written as

$$\begin{aligned}
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt} &= c_0 + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{j}{2\pi k} e^{j\omega_0 kt} \\
&= c_0 + \sum_{k=1}^{\infty}\left(\frac{-j}{2\pi k} e^{-j\omega_0 kt} + \frac{j}{2\pi k} e^{j\omega_0 kt}\right) \\
&= c_0 + \sum_{k=1}^{\infty}\left(\frac{1}{2\pi k} e^{-j\frac{\pi}{2}} e^{-j\omega_0 kt} + \frac{1}{2\pi k} e^{j\frac{\pi}{2}} e^{j\omega_0 kt}\right) \\
&= c_0 + \sum_{k=1}^{\infty}\left(\frac{1}{2\pi k} e^{-j(\omega_0 kt + \frac{\pi}{2})} + \frac{1}{2\pi k} e^{j(\omega_0 kt + \frac{\pi}{2})}\right) \\
&= c_0 + \sum_{k=1}^{\infty} \frac{1}{\pi k} \cos\left(\omega_0 t + \frac{\pi}{2}\right)
\end{aligned}$$

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5. (a) Show that the Fourier transform of $e^{-|t|}$ is $2/(1+\omega^2)$, and find the Fourier transform of the following signals:

$$x_1(t) = e^{-2|t|} \qquad x_2(t) = e^{-2|t|}\sin(t) \qquad x_3(t) = te^{-2|t|}.$$

(b) (4 marks) Use duality to find the Fourier transform of the signal

$$x(t) = \frac{1}{(1+t^2)}.$$

(10 marks)

(a) We are looking for the transform $X(\omega)$ of the signal

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t) = w(t) + w(-t),$$

where $w(t) = e^{-t}u(t)$. The tranform of $w(t)$ can be found in tranform tables

$$w(t) = e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} W(\omega)\frac{1}{1+j\omega},$$

and from time reversal

$$w(-t) = e^{t}u(-t) \overset{\mathcal{F}}{\longleftrightarrow} W(-\omega) = \frac{1}{1-j\omega}.$$

The resulting transform pair is therefore

$$e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}.$$

Using time scaling, $x_1(t) = e^{-2|t|} = x(2t)$, so

$$x_1(t) = e^{-2|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X\left(\frac{\omega}{2}\right) = \frac{1}{2}\frac{2}{1+(\omega/2)^2} = \frac{1}{1+(\omega/2)^2}.$$

Since

$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} j/2[X(\omega+\omega_0) - X(\omega-\omega_0)],$$

we have that

$$x_2(t) = e^{-2|t|}\sin(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j}{2}\left[\frac{1}{1+((\omega+1)/2)^2} - \frac{1}{1+((\omega-1)/2)^2}\right].$$

Finally, since $tx(t) \overset{\mathcal{F}}{\longleftrightarrow} jdX(\omega)/d\omega$,

$$x_3(t) = te^{-2|t|} \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\frac{1}{1+(\omega/2)^2} = \frac{-\omega/2}{[1+(\omega/2)^2]^2}$$

(b) From duality, $X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$, so

$$\frac{2}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}.$$

Therefore

$$\frac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}.$$

6. Find the transfer function of the system determined by the input/output relationship

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 12y = \frac{dx}{dt} + 2x,$$

and determine its impulse response.

(10 marks)

Taking the Fourier transform of both sides of the system expression gives

$$(j\omega)^2 Y(\omega) + 7(j\omega)Y(\omega) + 12Y(\omega) = (j\omega)X(\omega) + 2X(\omega)$$
$$\left[(j\omega)^2 + 7(j\omega) + 12\right] Y(\omega) = [(j\omega) + 2]\, X(\omega)$$
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(j\omega) + 2}{(j\omega)^2 + 7(j\omega) + 12}$$
$$H(\omega) = \frac{j\omega + 2}{(j\omega + 3)(j\omega + 4)}.$$

Doing a partial fraction expansion on the final expression above, we find that we can write $H(\omega)$ as

$$H(\omega) = \frac{-1}{(j\omega + 3)} + \frac{2}{(j\omega + 4)}.$$

Inverting using tables gives the impulse response

$$h(t) = -e^{-3t}u(t) + 2e^{-4t}u(t).$$

7. The signal $g(t)$ has the Fourier transform

$$G(\omega) = \frac{j\omega}{-\omega^2 + 5j\omega + 6}.$$

Find the Fourier transform of

(a) $g(2t)$

(b) $g(3t-6)$

(c) $g(-t)$

(d) $\frac{dx(t)}{dt}$

(e) $e^{-j100t}g(t)$.

(10 marks)

(a) From time scaling,

$$g(2t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}G(\omega/2) = \frac{j(\omega/2)}{-(\omega/2)^2 + 5j(\omega/2) + 6}.$$

(b) Since $g(3t-6) = g(3(t-2)) = h(t-2)$, with $h(t) = g(3t)$, we can see that

$$h(t) = g(3t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{3}G(\omega/3) = \frac{j(\omega/3)}{-(\omega/3)^2 + 5j(\omega/3) + 6}.$$

Time shift corresponds to multiplication by a complex exponential, so

$$g(3t-6) = h(t-2) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j(\omega/3)}{-(\omega/3)^2 + 5j(\omega/3) + 6} e^{-j\omega 2}.$$

(c) Time reversal says that $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-\omega)$, so

$$g(-t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j(-\omega)}{-(-\omega)^2 + 5j(-\omega) + 6} = \frac{j(-\omega)}{-\omega^2 + 5j(-\omega) + 6}.$$

(d) Since $dg(t)/dt \overset{\mathcal{F}}{\longleftrightarrow} j\omega G(\omega)$,

$$\frac{dg(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} \frac{(j\omega)^2}{-\omega^2 + 5j\omega + 6}.$$

(e) Since $e^{j\omega_0 t}g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(\omega - \omega_0)$,

$$e^{-j100t}g(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j(\omega+100)}{-(\omega+100)^2 + 5j(\omega+100) + 6}.$$

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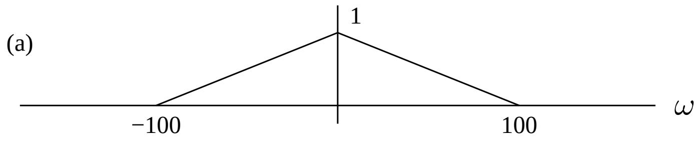
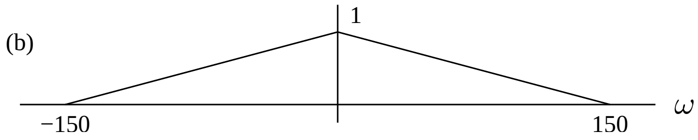
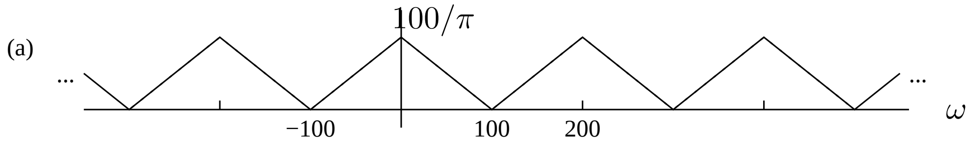
8. The signals with the frequency spectra shown below in (a) and (b) are sampled using an ideal sampler with $\omega_s = 200\text{rad}/s$. Sketch the frequency spectra of the sampled signals.

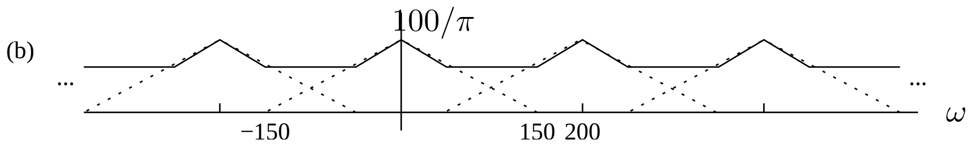




(10 marks)

A sampling frequency of $\omega_s = 200$rad/s corresponds to a sampling period of $T = \pi/100$s, which determines the scaling of the replicas in the Fourier transform of the sampled signal. The replicas themselves are at spacings of ω_s. Thus the spectra of the sampled signals are





INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\left(1 - \frac{2\|t\|}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$