EEE2047S: Signals and Systems I

Class Test

6 October 2021

SOLUTIONS

Please do not put your name or student number anywhere on this script.

Peoplesoft ID:

Information

- The test is closed-book.
- This test has *four* questions, totaling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 60 minutes.

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1. (5 marks) A modulator used in AM radio transmission has the input-output relationship

$$y(t) = x(t)\cos(\omega_c t),$$

where ω_c is a known and fixed carrier frequency. Determine whether the modulator system is (a) linear, (b) time invariant, and (c) causal.

(a) Suppose $x_1(t) \longrightarrow y_1(t) = x_1(t)\cos(\omega_c t)$ and $x_2(t) \longrightarrow y_2(t) = x_2(t)\cos(\omega_c t)$ are valid input-output pairs for the system. Consider now the input $x_3(t) = ax_1(t) + bx_2(t)$. The output is then

$$\begin{aligned} y_3(t) &= x_3(t)\cos(\omega_c t) = (ax_1(t) + bx_2(t))\cos(\omega_c t) \\ &= ax_1(t)\cos(\omega_c t) + bx_2(t)\cos(\omega_c t) = ay_1(t) + by_2(t) \end{aligned}$$

so the system is linear.

(b) Suppose $x_1(t) \longrightarrow y_1(t) = x_1(t)\cos(\omega_c t)$ and consider the input $x_4(t) = x_1(t-c)$. The output is then

$$y_4(t) = x_4(t)\cos(\omega_c t) = x_1(t-c)\cos(\omega_c t).$$

In general $y_4(t) \neq y_1(t-c)$ so the system is not time invariant.

(c) To calculate the output at say $t = 10$ one only needs to know the input at time $t = 10$, so the system is causal (and also memoryless).

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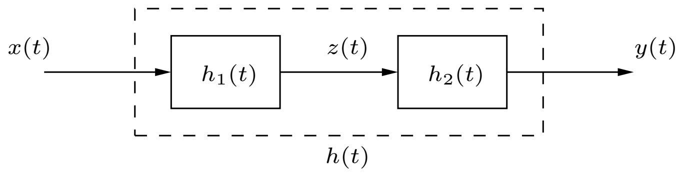
2. (5 marks) Consider two systems connected as shown below,



where the impulse responses of the two systems are $h_1(t) = u(t)$ and

$$h_2(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and sketch the effective impulse response $h(t)$ linking the input $x(t)$ to the output $y(t)$.

Since $z(t) = h_1(t) * x(t)$ and $y(t) = h_2(t) * z(t)$ we see that

$$y(t) = h_2(t) * h_1(t) * x(t) = [h_2(t) * h_1(t)] * x(t) = h(t) * x(t),$$

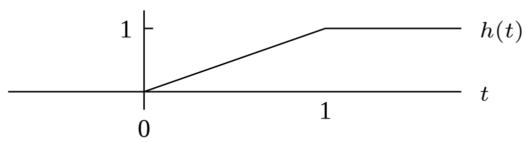
where the effective impulse response is $h(t) = h_2(t) * h_1(t)$.
Now $h_2(t) = u(t) - u(t-1)$ and

$$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \int_{0}^{\infty} u(t-\tau)d\tau = tu(t),$$

so

$$\begin{aligned} h(t) &= (u(t) - u(t-1)) * u(t) = u(t) * u(t) - u(t-1) * u(t) \\ &= tu(t) - (t-1)u(t-1) = tu(t) - tu(t-1) + u(t-1) \\ &= t(u(t) - u(t-1)) + u(t-1). \end{aligned}$$

Plot follows:



3. (5 marks) The signal $f(t) = -2\cos(\omega_0 t) + 2\sqrt{3}\sin(\omega_0 t + \pi/3)$ can be written in the form

$$f(t) = ce^{j\omega_0 t} + c^* e^{-j\omega_0 t},$$

where c^* is the conjugate of c. Find the value of c in this representation. Recall Euler's formula $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

The result follows from simple expansion in terms of complex exponentials:

$$\begin{aligned} f(t) &= -2\cos(\omega_0 t) + 2\sqrt{3}\sin(\omega_0 t + \pi/3) \\ &= -\frac{2}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) + 2\sqrt{3}\frac{1}{2j}(e^{j(\omega_0 t + \pi/3)} - e^{-j(\omega_0 t + \pi/3)}) \\ &= \left(-1 + \frac{\sqrt{3}}{j}e^{j\pi/3}\right)e^{j\omega_0 t} + \left(-1 - \frac{\sqrt{3}}{j}e^{-j\pi/3}\right)e^{-j\omega_0 t}, \end{aligned}$$

thus $c = -1 + (\sqrt{3}/j)e^{j\pi/3}$.

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4. (5 marks)

(a) Find the Fourier transform of the triangular pulse signal defined as follows:

$$x_1(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 1-t & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find and sketch the Fourier transform of the signal

$$x_2(t) = 2\text{sinc}^2(3t).$$

(a) Noting that

$$x_1(t) = \left(1 - \frac{2|t|}{\tau}\right) p_\tau(t)$$

for $\tau = 2$ we can read the required result from Fourier tables:

$$X_1(\omega) = \text{sinc}^2\left(\frac{\omega}{2\pi}\right).$$

(b) Using the transform pair

$$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau t}{4\pi}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\left(1 - \frac{2|\omega|}{\tau}\right) p_\tau(\omega)$$

with $\tau = 12\pi$ we see that

$$6\pi\text{sinc}^2(3t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\left(1 - \frac{|\omega|}{6\pi}\right) p_{12\pi}(\omega)$$

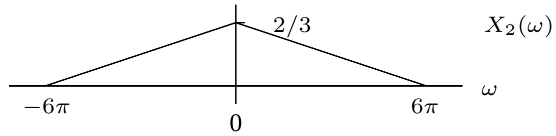
so

$$2\text{sinc}^2(3t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2}{3}\left(1 - \frac{|\omega|}{6\pi}\right) p_{12\pi}(\omega).$$

The Fourier domain representation of $x_2(t)$ is therefore

$$X_2(\omega) = \frac{2}{3}\left(1 - \frac{|\omega|}{6\pi}\right) p_{12\pi}(\omega) = \begin{cases} \frac{2}{3}\left(1 - \frac{|\omega|}{6\pi}\right) & |\omega| < 6\pi \\ 0 & \text{otherwise.} \end{cases}$$

Plot follows:



INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c}$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b}$ ($b > 0$)
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$ and $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(s) * V(s)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$
Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $b = \sqrt{c-a^2}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$