EEE2047S: Signals and Systems I

Class Test 1

14 August 2019

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totaling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 60 minutes.

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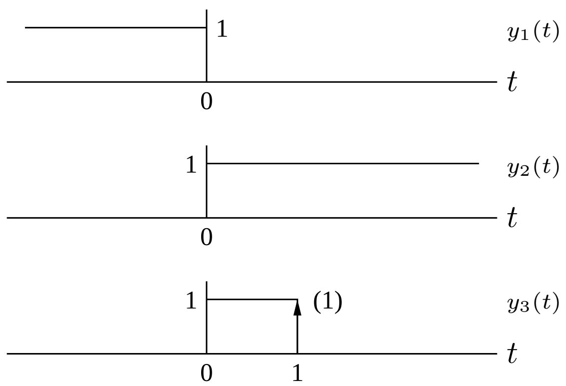
1. (5 marks) Plot the following signals:
   (a) $y_1(t) = \int_t^\infty \delta(\lambda) d\lambda$
   (b) $y_2(t) = \int_0^\infty \delta(\lambda - t) d\lambda$
   (c) $y_3(t) = \frac{d}{dt} \left(t(u(t) - u(t-1))\right)$.

   (a) If $t > 0$ then the integration limits do not include the delta function at the origin, so the output is zero. If $t < 0$ then the limits do include the delta function, so the integral is the area, namely one. Thus $y_1(t) = u(-t)$.
   (b) The integrand consists of a delta function at $\lambda = t$. For $t > 0$ the delta function is in the range of the integral and the result is one. For $t < 0$ the delta function is not included in the integration range and the result is zero. Thus $y_2(t) = u(t)$.
   (c) If we consider the quantity $t(u(t) - u(t-1))$ we find that

$$t(u(t) - u(t-1)) = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   The required quantity is the generalised derivative:
   $y_3(t) = (u(t) - u(t-1)) + \delta(t-1)$.

   Plots as follows:



2. (5 marks) Suppose the output $y(t)$ of a system is related to the input $x(t)$ via the relation

$$y(t) = 3x(t+2) + 4x(t-1).$$

(a) Is the system causal? Why?

(b) Find and plot the output $y(t)$ when the input is $x(t) = u(t-1)$.

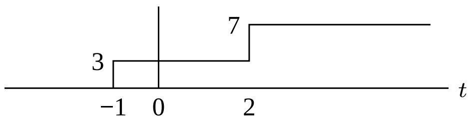
(c) Is the system time invariant? Why?

(a) To calculate the output at say $t = 10$ one needs to know the input at $t = 10 - 1 = 9$ and at $t = 10 + 2 = 12$. This latter instant is in the future, so the system is not causal.

(b) For $x(t) = u(t-1)$ the output is

$$y(t) = 3x(t+2) + 4x(t-1) = 3u(t+1) + 4u(t-2).$$

Plot below:



(c) Suppose the input $x_1(t)$ generates the output $y_1(t)$, and $x_2(t)$ generates $y_2(t)$. Then we must have

$$y_1(t) = 3x_1(t+2) + 4x_1(t-1) \quad \text{and} \quad y_2(t) = 3x_2(t+2) + 4x_2(t-1).$$

If in addition $x_2(t) = x_1(t-c)$ then we see that

$$\begin{aligned} y_2(t) &= 3x_2(t+2) + 4x_2(t-1) = 3x_1(t+2-c) + 4x_2(t-1-c) \\ &= 3x_1((t-c)+2) + 4x_2((t-c)-1) = y_1(t-c). \end{aligned}$$

Thus a shift by c in the input leads to a corresponding shift by c in the output, so the system is time invariant.

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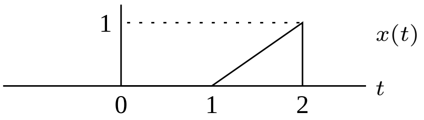
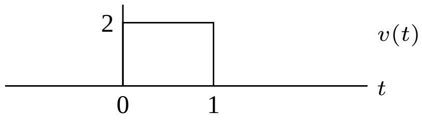
3. (5 marks) Consider the signal $y(t) = x(t) * v(t)$ for the signals given below:





(a) For what range of values of t is $y(t) = 0$?

(b) Specify $y(t)$ over the interval $1 \leq t \leq 2$.

(a) The convolution is given by

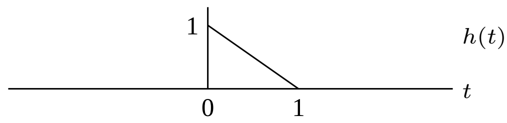
$$y(t) = \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)d\lambda.$$

Since $v(t-\lambda)$ is nonzero for $\lambda = t-1$ to $\lambda = t$ we can see from graphical means that $y(t) = 0$ for $t < 1$ and for $t - 1 > 2$ (i.e. $t > 3$).

(b) Again using graphical means we find that for $1 \leq t \leq 2$ the result is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)d\lambda = \int_{1}^{t} (\lambda - 1)(2)d\lambda = 2\left[\lambda^2/2 - \lambda\right]_{\lambda=1}^{t} \\ &= 2((t^2/2 - t) - (1/2 - 1)) = 2(t^2/2 - t + 1/2) = (t^2 - 2t + 1) \\ &= (t-1)^2. \end{aligned}$$

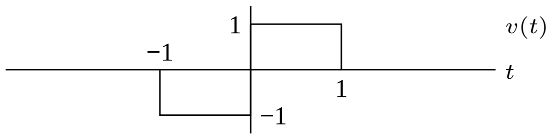
4. (5 marks) Consider an LTI system with impulse response



(a) Is the system causal? Why?

(b) Find and sketch the step response of the system, i.e. the output $g(t)$ when the input is $x(t) = u(t)$.

(c) Find the response of the system to the signal below:



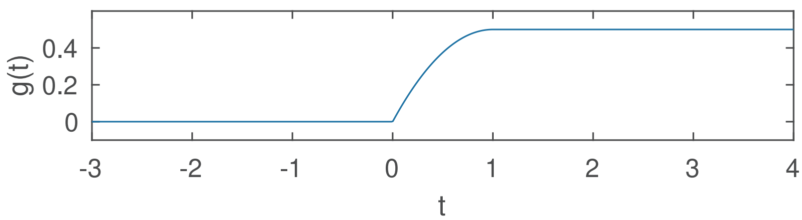
Express the result in terms of the step response $g(t)$ in (b), and sketch your solution.

(a) The impulse response is right-sided (zero for $t < 0$) so the system is causal.

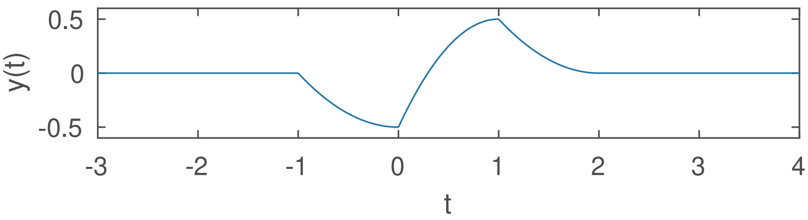
(b) Applying the integration property to the pair $\delta(t) \longrightarrow h(t)$ gives the step response

$$g(t) = \int_{-\infty}^{t} h(\tau)d\tau = \begin{cases} 0 & t \leq 0 \\ \int_0^t (1-\tau)d\tau & 0 < t < 1 \\ \int_0^1 (1-\tau)d\tau & t \geq 1. \end{cases}$$
$$= (t - t^2/2)(u(t) - u(t-1)).$$

Plot as follows:



(c) The input can be written as $v(t) = -u(t+1) + 2u(t) - u(t-1)$. Since $u(t) \longrightarrow g(t)$, using linearity and time invariance the required output is $y(t) = -g(t+1) + 2g(t) - g(t-1)$. Plot as follows:



0.5
0
-0.5
y(t)
-3
-2
-1
0
1
2
3
4
t

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \operatorname{sinc}\frac{\tau\omega}{2\pi}$
$\tau \operatorname{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\operatorname{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t+\tau/2) - u(t-\tau/2)$ and $\operatorname{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(s) * V(s)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$
Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $b = \sqrt{c-a^2}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$