EEE2047S: Signals and Systems I

Class Test 2

10 October 2018

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book. You are welcome and encouraged to have some blank sheets of paper to do roughwork on.
- This test has *four* questions, totaling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 45 minutes.

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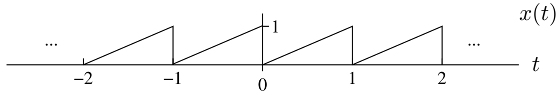
1. (5 marks) The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

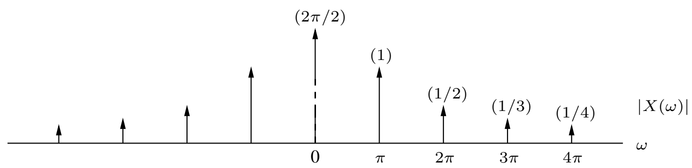
$$c_k = \begin{cases} 1/2 & k = 0 \\ -\frac{1}{jk2\pi} & \text{otherwise.} \end{cases}$$

Suppose $X(\omega)$ is the Fourier transform of $x(t)$. Find and plot the magnitude $|X(\omega)|$ over the range $-4\pi \leq \omega \leq 4\pi$.

Using the Fourier pair $e^{j\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$ we find that

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\pi).$$

Now $|c_k| = 1/(2\pi k)$, so the required signal is a train of delta functions of appropriate sizes:



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2. (5 marks) Use the "Multiplication by power of t" property along with the Fourier pair

$$e^{-bt}u(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{j\omega + b} \qquad (b > 0)$$

to find the inverse transform of

$$X(\omega) = \frac{10}{(j\omega + 4)^2}.$$

Applying the property

$$tx(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad j\frac{d}{d\omega}X(\omega)$$

to the given pair yields the new Fourier pair

$$te^{-bt}u(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad j\frac{d}{d\omega}(j\omega + b)^{-1} = -j(j)(j\omega + b)^{-2} = \frac{1}{(j\omega + b)^2}.$$

Using linearity we therefore have $x(t) = 10te^{-4t}u(t)$.

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3. (5 marks) Find the Fourier transform of the signal

$$x(t) = te^{-2(t-1)}\cos(4t)u(t).$$

The signal can be written as

$$x(t) = e^2te^{-2t}u(t)\cos(4t).$$

Start with

$$e^{-2t}u(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{2 + j\omega}.$$

Applying the "multiplication by power of t" property gives

$$te^{-2t}u(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}(2 + j\omega)^{-1} = \frac{1}{(2 + j\omega)^2}.$$

The modulation property applied to this gives

$$te^{-2t}u(t)\cos(4t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\frac{1}{(2 + j(\omega - 4))^2} + \frac{1}{2}\frac{1}{(2 + j(\omega + 4))^2}.$$

Finally linearity gives the required transform

$$X(\omega) = \frac{e^2}{2}\frac{1}{(2 + j(\omega - 4))^2} + \frac{e^2}{2}\frac{1}{(2 + j(\omega + 4))^2}.$$

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4. (5 marks) The signal $x(t) = 5\sin(10t + \pi/4)$ is input to a LTI system with frequency response $H(\omega) = \frac{10}{10+j\omega}$. The output signal can be written in the form

$$y(t) = Ae^{j10t} + A^* e^{-j10t}.$$

Specify the value of A.

We can write the signal as

$$x(t) = \frac{5}{2j}(e^{j(10t+\pi/4)} - e^{-j(10t+\pi/4)}) = \frac{5}{2j}e^{j\pi/4}e^{j10t} - \frac{5}{2j}e^{-j\pi/4}e^{-j10t}$$

and the frequency response is

$$\begin{aligned} y(t) &= \frac{5}{2j}e^{j\pi/4}H(10)e^{j10t} - \frac{5}{2j}e^{-j\pi/4}H(-10)e^{-j10t}. \\ &= \frac{5}{2j}e^{j\pi/4}\frac{10}{10+j(10)}e^{j10t} - \frac{5}{2j}e^{-j\pi/4}\frac{10}{10+j(-10)}e^{-j10t} \\ &= Ae^{j10t} + A^* e^{-j10t} \end{aligned}$$

with

$$A = \frac{5}{2j}e^{j\pi/4}\frac{10}{10+j(10)} = -1.7678j = 1.7678e^{-j\pi/2}.$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau\,\text{sinc}\frac{\tau\omega}{2\pi}$
$\tau\,\text{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right)p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right)p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t+\tau/2) - u(t-\tau/2)$ and $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(s) * V(s)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$

Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta - br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $b = \sqrt{c-a^2}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$