EEE2047S: Signals and Systems I

Class Test 1

27 August 2018

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totaling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Evaluate the following integrals:
   (a) $\int_{-\infty}^{\infty} \cos(5\pi t)\delta(t)dt$
   (b) $\int_{5}^{\infty} \cos(5\pi t)\delta(t)dt$
   (c) $\int_{-\infty}^{\infty} \cos(5\pi(t-\lambda))\delta(t)dt$
   (d) $\int_{-\infty}^{\infty} \cos(5\pi(t-\lambda))\delta(\lambda)d\lambda$
   (e) $\int_{0}^{\infty} \cos(5\pi t)\delta(t-\lambda)d\lambda$

(a) Result as follows:

$$\int_{-\infty}^{\infty} \cos(5\pi t)\delta(t)dt = \int_{-\infty}^{\infty} \cos(5\pi 0)\delta(t)dt = \cos(0)\int_{-\infty}^{\infty} \delta(t)dt = 1(1) = 1.$$

(b) Result as follows:

$$\int_{5}^{\infty} \cos(5\pi t)\delta(t)dt = \int_{5}^{\infty} \cos(5\pi 0)\delta(t)dt = \cos(0)\int_{5}^{\infty} \delta(t)dt = 1(0) = 0.$$

(c) Result as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(5\pi(t-\lambda))\delta(t)dt &= \int_{-\infty}^{\infty} \cos(5\pi(0-\lambda))\delta(t)dt \\ &= \cos(5\pi(0-\lambda))\int_{-\infty}^{\infty} \delta(t)dt = \cos(-5\pi\lambda).\end{aligned}$$

(d) Result as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(5\pi(t-\lambda))\delta(\lambda)d\lambda &= \int_{-\infty}^{\infty} \cos(5\pi(t-0))\delta(\lambda)d\lambda \\ &= \cos(5\pi(t-0))\int_{-\infty}^{\infty} \delta(\lambda)d\lambda = \cos(5\pi t).\end{aligned}$$

(e) Result as follows:

$$\int_{0}^{\infty} \cos(5\pi t)\delta(t-\lambda)d\lambda = \cos(5\pi t)\int_{0}^{\infty} \delta(t-\lambda)d\lambda = \cos(5\pi t)u(t).$$

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2. (5 marks) Consider a system where the input $x(t)$ and output $y(t)$ are related by the following:

$$y(t) = x(t^2).$$

(a) Is the system linear? Justify.

(b) Is the system causal? Justify.

(c) Find and plot the output for the two cases where $x(t) = u(t)$ and $x(t) = u(t-1)$.

(d) Is the system time invariant? Justify.

(a) Suppose the response to $x_1(t)$ is $y_1(t)$ and the response to $x_2(t)$ is $y_2(t)$. Then $y_1(t) = x_1(t^2)$ and $y_2(t) = x_2(t^2)$. If the input is $x(t) = ax_1(t) + bx_2(t)$ then the corresponding output will be

$$y(t) = x(t^2) = ax_1(t^2) + bx_2(t^2) = ay_1(t) + by_2(t),$$

so the system is linear.

(b) To calculate the output $y(t)$ at time $t = 2$ one needs to know $x(t)$ at time $t = 2^2 = 4$, which is in the future. Thus the system is not causal.

(c) When the input is $x_1(t) = u(t)$ then the output will be $y_1(t) = x_1(t^2) = u(t^2) = 1$ since t^2 is always nonnegative. For $x_2(t) = u(t-1)$ the output will be

$$y_2(t) = x_2(t^2) = u(t^2 - 1) = \begin{cases} 1 & t^2 - 1 \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

or

$$y_2(t) = \begin{cases} 1 & |t| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

These should be plotted.

(d) We observe from the previous question that $x_2(t) = x_1(t-1)$ but $y_2(t) \neq y_1(t-1)$, so the system is not time invariant. Many other counterexamples will lead to the same conclusion.

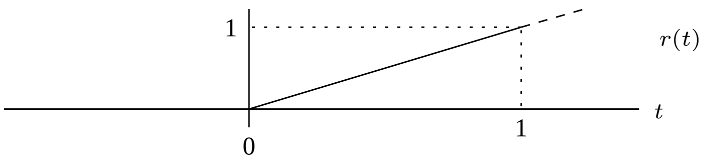
3. (5 marks) Suppose we define the right-sided ramp signal as $r(t) = tu(t)$, where $u(t)$ is the unit step. Then:

   (a) Sketch $r(t)$

   (b) Show that $u(t) * u(t+1) = r(t+1)$, where $u(t)$ is the unit step.

   (c) Use the previous result to find $y(t) = u(t) * p_1(t - 1/2)$.

(a) Sketch is below:



(b) Let $w(t) = u(t) * u(t)$. Then $\dot{w}(t) = u(t) * \dot{u}(t) = u(t) * \delta(t) = u(t)$. Thus

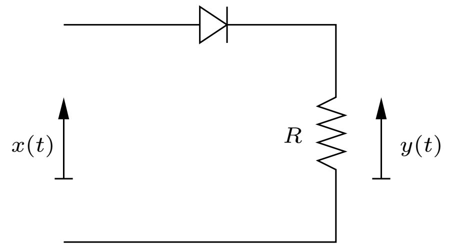
$$w(t) = \int_{-\infty}^{t} \dot{w}(\tau) d\tau = \int_{-\infty}^{t} u(\tau) d\tau = r(t).$$

Shifting one of the inputs to a convolution causes a corresponding shift in the output, thus $u(t) * u(t+1) = w(t+1) = r(t+1)$.

(c) Since

$$\begin{aligned} y(t) &= u(t) * p_1(t - 1/2) = u(t) * (u(t) - u(t-1)) \\ &= u(t) * u(t) - u(t) * u(t-1) = r(t) - r(t-1). \end{aligned}$$

4. (5 marks) The half-wave rectifier circuit below is driven by a voltage signal $x(t)$, and the output voltage across the resistor is $y(t)$:



Assume an ideal diode with zero forward voltage drop.

(a) Sketch the output $y(t)$ when the input is $x(t) = \sin(t)$.

(b) What is the fundamental period of $y(t)$?

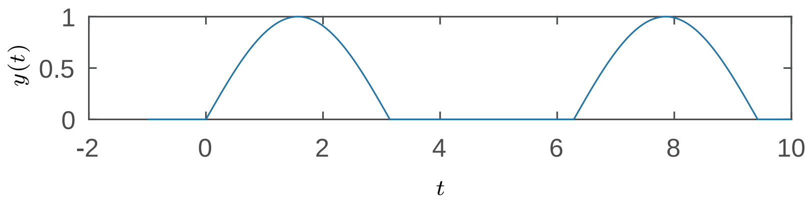
(c) What is the average power of $y(t)$?

(d) The output can be written in the form

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt}.$$

Find the value of c_1 in this representation, expressed in polar form.

(a) When $x(t) < 0$ the diode will not conduct, there will be no current through the resistor, and the output $y(t)$ will be zero. When $x(t) > 0$ current will flow and the full input voltage will appear across R, so $y(t) = x(t)$. Thus $y(t)$ is as follows:



(b) The fundamental period is $T = 2\pi$ radians per second.

(c) The average power can be found as follows:

$$P_{\text{av}} = \frac{1}{T}\int_0^T |y(t)|^2 dt = \frac{1}{2\pi}\int_0^\pi \sin^2(t) dt = \frac{1}{2\pi}\int_0^\pi \frac{1}{2}(1-\cos(2t))dt$$

$$= \frac{1}{4} - \frac{1}{4\pi}\left[\frac{1}{2}\sin(2t)\right]_{t=0}^{\pi} = 1/4 \text{ Watts.}$$

(d) Since $\omega_0 = 2\pi/T = 1$ the coefficient can be calculated as follows:

$$\begin{aligned} c_1 &= \frac{1}{2\pi}\int_0^{\pi} \sin(t)e^{-jt}dt = \frac{1}{2\pi}\int_0^{\pi} \frac{1}{2j}(e^{jt} - e^{-jt})e^{-jt}dt \\ &= \frac{1}{4\pi j}\int_0^{\pi} dt - \frac{1}{4\pi j}\int_0^{\pi} e^{-j2t}dt = \frac{1}{4j} = \frac{1}{4}e^{-j\pi/2}. \end{aligned}$$
