EEE2047S: Signals and Systems I

Class Test 1

18 October 2017

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book. You are welcome and encouraged to have some blank sheets of paper to do roughwork on.
- This test has *four* questions, totaling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 45 minutes.

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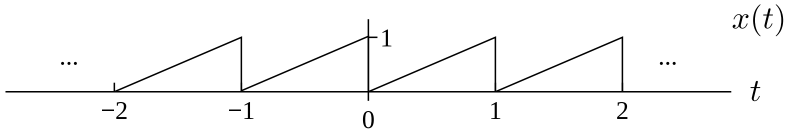
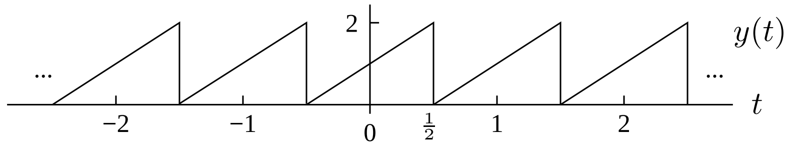
1. (5 marks) The periodic signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi t} \qquad \text{with} \qquad c_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{-1}{jk2\pi} & k \neq 0. \end{cases}$$

Suppose $y(t)$ is the signal below:



It has a Fourier series representation $y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk2\pi t}$. Show that $d_k = 2c_k e^{-jk\pi}$, and plot the coefficient magnitudes $|d_k|$ over the range $k = -4, \ldots, 4$.

(Bonus mark) Suppose $Y(\omega)$ is the Fourier transform of $y(t)$. Find and plot the magnitude $|Y(\omega)|$ over the range $-4\pi \leq \omega \leq 4\pi$.

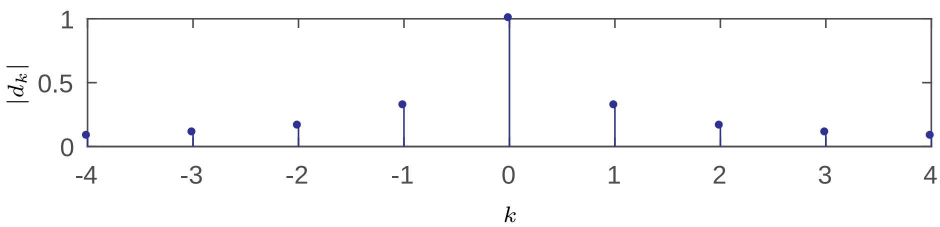
The signals are related by $y(t) = 2x(t - 1/2)$ so

$$y(t) = 2 \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi(t-1/2)} = \sum_{k=-\infty}^{\infty} 2c_k e^{-jk2\pi/2} e^{jk2\pi t} = \sum_{k=-\infty}^{\infty} d_k e^{j2k\pi t}$$

with $d_k = 2c_k e^{-jk\pi}$. Thus for $k \neq 0$ we have

$$|d_k| = 2|c_k| = 2\left|\frac{-1}{jk2\pi}\right| = \left|\frac{j}{k\pi}\right| = \left|\frac{1}{k\pi} e^{j\pi/2}\right| = \frac{1}{|k\pi|},$$

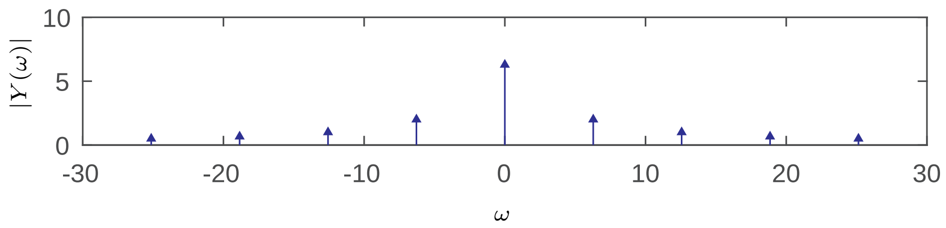
and $|d_0| = 2|c_0| = 1$. The plot follows:



The Fourier transform of $y(t)$ is

$$Y(\omega) = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} d_k e^{jk2\pi t}\right\} = \sum_{k=-\infty}^{\infty} d_k \mathcal{F}\left\{e^{jk2\pi t}\right\} = \sum_{k=-\infty}^{\infty} 2\pi d_k \delta(\omega - 2\pi k),$$

and the plot follows:



Note the this can be generated from the previous plot. There are impulses at $\omega = 2\pi k$ and the weights of these impulses are 2π times the corresponding coefficients $|c_k|$.

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2. (5 marks) Find the inverse Fourier transform of

$$X(\omega) = e^{-2j\omega}(u(\omega) - u(\omega - 2)).$$

Noting that $u(\omega) - u(\omega - 2) = p_2(\omega - 1)$ we have $X(\omega) = e^{-2j\omega} p_2(\omega - 1)$. Applying frequency shift with $\omega_0 = 1$ to the pair

$$\tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi p_\tau(\omega)$$

with $\tau = 2$ gives

$$2 \operatorname{sinc}\left(\frac{t}{\pi}\right) e^{jt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi p_2(\omega - 1),$$

and by linearity

$$\frac{1}{\pi} \operatorname{sinc}\left(\frac{t}{\pi}\right) e^{jt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad p_2(\omega - 1),$$

Time shift with $c = 2$ then gives the pair

$$\frac{1}{\pi} \operatorname{sinc}\left(\frac{t-2}{\pi}\right) e^{j(t-2)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad e^{-2j\omega} p_2(\omega - 1),$$

so the inverse is

$$x(t) = \frac{1}{\pi} \operatorname{sinc}\left(\frac{t-2}{\pi}\right) e^{j(t-2)}.$$

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3. (5 marks) Suppose G is a causal system described by the differential equation

$$\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t) - x(t).$$

(a) Show that the frequency response of G is

$$H(\omega) = \frac{j\omega - 1}{j\omega + 1}.$$

(b) Find the impulse response of G.

(Bonus mark) Find the step response of G, or the output when $x(t) = u(t)$ under initial rest conditions.

(a) Since

$$Y(\omega)(j\omega + 1) = X(\omega)(j\omega - 1)$$

we have

$$H(\omega) = \frac{j\omega - 1}{j\omega + 1}.$$

(b) We have

$$H(\omega) = \frac{j\omega}{j\omega + 1} - \frac{1}{j\omega + 1}.$$

Applying the time differentiation property to $e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} 1/(j\omega + 1)$ gives the pair

$$\frac{d}{dt}e^{-t}u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{j\omega}{j\omega + 1},$$

so the impulse response is

$$h(t) = \frac{d}{dt}\left\{e^{-t}u(t)\right\} - e^{-t}u(t) = \delta(t) - e^{-t}u(t) - e^{-t}u(t) = \delta(t) - 2e^{-t}u(t).$$

The step response is $g(t) = \int_{-\infty}^{t}[\delta(\lambda) - 2e^{-2\lambda}u(\lambda)]d\lambda$. For $t < 0$ we have $g(t) = 0$, and for $t \geq 0$

$$g(t) = u(t) - 2\int_0^t e^{-\lambda}d\lambda = 1 - 2[-e^{-\lambda}]_{\lambda=0}^{t} = 1 - 2(1 - e^{-t}),$$

so $g(t) = u(t) - 2(1 - e^{-t})u(t)$.

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4. (5 marks) An LTI system has a frequency response

$$H(\omega) = e^{j5\omega} \frac{1}{4 + j\omega}.$$

(a) Find the impulse response of the system.

(b) When the input is

$$x(t) = \sin(4t)$$

then the output can be written in the form $y(t) = ce^{j\omega_0 t} + c^* e^{-j\omega_0 t}$ for some complex value c. Specify the value of c in polar form, and the value of ω_0.

(a) The impulse response is the inverse transform. Applying the time shift property to the pair $e^{-4t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} 1/(4 + j\omega)$ gives

$$e^{-4(t+5)}u(t+5) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad e^{j5\omega} 1/(4 + j\omega),$$

so the impulse response is $h(t) = e^{-4(t+5)}u(t+5)$.

(b) We can write the input as

$$x(t) = \sin(4t) = \frac{1}{2j} e^{j4t} - \frac{1}{2j} e^{-j4t}.$$

The output from the system will be

$$y(t) = \frac{1}{2j} H(4) e^{j4t} - \frac{1}{2j} H(-4) e^{-j4t}.$$

Thus we have coefficient

$$c = \frac{1}{2j} H(4) = e^{j5(4)} \frac{1}{4 + j4} \frac{1}{2j} = e^{j20} \frac{1}{-8 + j8} = e^{j20} \frac{1}{8\sqrt{2} e^{j3\pi/4}} = \frac{1}{8\sqrt{2}} e^{j(20 - 3\pi/4)}$$

and $\omega_0 = 4$.

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \Longleftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \Longleftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \Longleftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \Longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \Longleftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \Longleftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \Longleftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

with $p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$ and $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$.

Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \Longleftrightarrow aX(s) + bV(s)$
Time shift	$x(t-a)u(t-a) \Longleftrightarrow e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \Longleftrightarrow \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \Longleftrightarrow (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \Longleftrightarrow X(s-a)$
Differentiation	$x'(t) \Longleftrightarrow sX(s) - x(0^-)$
	$x''(t) \Longleftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$
	$x^{(n)}(t) \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\lambda)d\lambda \Longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \Longleftrightarrow X(s)V(s)$
Frequency convolution	$x(t)v(t) \Longleftrightarrow \frac{1}{2\pi j}X(\omega) * V(\omega)$

Initial value: $f(0^+) = \lim_{s\to\infty} sF(s)$

Final value: $f(\infty) = \lim_{s\to 0} sF(s)$ with all poles in left-hand plane

Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at}\cos(bt+\theta)u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $b = \sqrt{c-a^2}$, $\theta = \tan^{-1}\frac{Aa-b}{A\sqrt{c-a^2}}$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$