

# EEE2047S: Signals and Systems I

## Class Test 1

18 October 2017

Name:

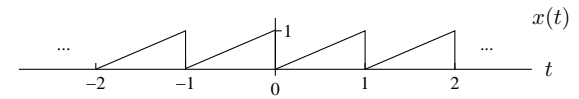
Student number:

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### Information

- The test is closed-book. You are welcome and encouraged to have some blank sheets of paper to do roughwork on.
  - This test has *four* questions, totaling 20 marks.
  - There is an information sheet attached at the end of this paper.
  - Answer *all* the questions.
  - You have 45 minutes.
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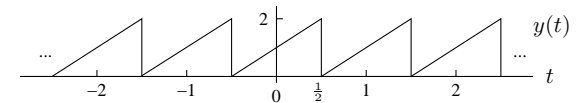
1. (5 marks) The periodic signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi t} \quad \text{with} \quad c_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{-1}{jk2\pi} & k \neq 0. \end{cases}$$

Suppose  $y(t)$  is the signal below:



It has a Fourier series representation  $y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk2\pi t}$ . Show that  $d_k = 2c_k e^{-jk\pi}$ , and plot the coefficient magnitudes  $|d_k|$  over the range  $k = -4, \dots, 4$ .

(Bonus mark) Suppose  $Y(\omega)$  is the Fourier transform of  $y(t)$ . Find and plot the magnitude  $|Y(\omega)|$  over the range  $-4\pi \leq \omega \leq 4\pi$ .

2. (5 marks) Find the inverse Fourier transform of

$$X(\omega) = e^{-2j\omega}(u(\omega) - u(\omega - 2)).$$

3. (5 marks) Suppose  $G$  is a causal system described by the differential equation

$$\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t) - x(t).$$

- (a) Show that the frequency response of  $G$  is

$$H(\omega) = \frac{j\omega - 1}{j\omega + 1}.$$

- (b) Find the impulse response of  $G$ .

(Bonus mark) Find the step response of  $G$ , or the output when  $x(t) = u(t)$  under initial rest conditions.

4. (5 marks) An LTI system has a frequency response

$$H(\omega) = e^{j5\omega} \frac{1}{4 + j\omega}.$$

(a) Find the impulse response of the system.

(b) When the input is

$$x(t) = \sin(4t)$$

then the output can be written in the form  $y(t) = ce^{j\omega_0 t} + c^*e^{-j\omega_0 t}$  for some complex value  $c$ . Specify the value of  $c$  in polar form, and the value of  $\omega_0$ .

## INFORMATION SHEET

### Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \iff aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \iff X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \iff \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \iff X(-\omega) = \overline{X(\omega)}$
Multiplication by power of $t$	$t^n x(t) \iff j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Frequency shift	$x(t)e^{j\omega_0 t} \iff X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \iff \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \iff (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda)d\lambda \iff \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \iff X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \iff \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^2 d\omega$
Duality	$X(t) \iff 2\pi x(-\omega)$

### Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$
1 $(-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad (c \text{ any real number})$
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad (\omega_0 \text{ any real number})$
$p_{\tau}(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_{\tau}(\omega)$
$\left(1 - \frac{2 t }{\tau}\right)p_{\tau}(t)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2 \omega }{\tau}\right)p_{\tau}(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

with  $p_{\tau}(t) = u(t + \tau/2) - u(t - \tau/2)$  and  $\text{sinc}(\lambda) = \sin(\pi\lambda)/(\pi\lambda)$ .

# Laplace transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \iff aX(s) + bV(s)$
Time shift	$x(t - a)u(t - a) \iff e^{-as}X(s) \quad a \geq 0$
Time scaling	$x(at) \iff \frac{1}{a}X(\frac{s}{a}) \quad a > 0$
Frequency differentiation	$t^n x(t) \iff (-1)^n X^{(n)}(s)$
Frequency shift	$e^{at}x(t) \iff X(s - a)$
Differentiation	$x'(t) \iff sX(s) - x(0^-)$ $x''(t) \iff s^2X(s) - sx(0^-) - x'(0^-)$ $x^{(n)}(t) \iff s^nX(s) - s^{n-1}x(0^-) - \dots - x^{(n-1)}(0^-)$
Integration	$\int_{0^-}^t x(\lambda)d\lambda \iff \frac{1}{s}X(s)$ $\int_{-\infty}^t x(\lambda)d\lambda \iff \frac{1}{s}X(s) + \frac{1}{s} \int_{-\infty}^0 x(\lambda)d\lambda$
Time convolution	$x(t) * v(t) \iff X(s)V(s)$
Frequency convolution	$x(t)v(t) \iff \frac{1}{2\pi j}X(\omega) * V(\omega)$

Initial value:  $f(0^+) = \lim_{s \rightarrow \infty} sF(s)$   
Final value:  $f(\infty) = \lim_{s \rightarrow 0} sF(s)$  with all poles in left-hand plane

# Common Unilateral Laplace Transform Pairs

$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$	$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{(r \cos \theta)s + (ar \cos \theta - br \sin \theta)}{s^2 + 2as + (a^2 + b^2)}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{0.5re^j\theta}{s+a-jb} + \frac{0.5re^{-j\theta}}{s+a+jb}$
$re^{-at} \cos(bt + \theta)u(t)$	$\frac{As+B}{s^2+2as+c}$ $r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}, b = \sqrt{c-a^2}, \theta = \tan^{-1} \frac{Aa-b}{A\sqrt{c-a^2}}$

# Trigonometric identities

$$\begin{aligned} \sin(-\theta) &= -\sin(\theta) & \cos(-\theta) &= \cos(\theta) & \tan(-\theta) &= -\tan(\theta) & \sin^2(\theta) + \cos^2(\theta) &= 1 \\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta) & \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ e^{j\theta} &= \cos(\theta) + j\sin(\theta) \end{aligned}$$