

# EEE2035F: Signals and Systems I

Class Test 2

25 April 2016

## SOLUTIONS

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Name:

Student number:

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### Information

- The test is closed-book.
  - This test has *four* questions, totalling 20 marks.
  - Answer *all* the questions.
  - You have 45 minutes.
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1. (5 marks) Consider the signal

$$x(t) = \sin\left(3\pi t + \frac{\pi}{3}\right).$$

(a) Show that the signal can be written as a complex exponential Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(3\pi)t}$$

with  $c_1 = c_{-1}^* = \frac{1}{2j} e^{j\pi/3}$  and all other coefficients zero. Plot the magnitude and phase of the coefficients  $c_k$  as a function of  $k$ .

(b) The signal  $x(t)$  can also be expressed as a standard trigonometric Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t + b_k).$$

Specify  $\omega_0$  and the coefficients  $a_k$  and  $b_k$ .

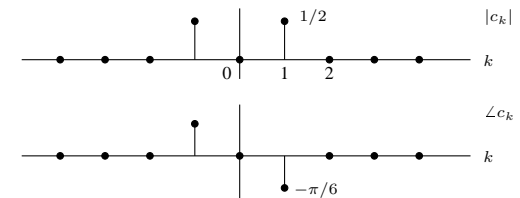
(a) Using Euler's expansion we get

$$x(t) = \frac{1}{2j} (e^{j(3\pi t + \pi/3)} - e^{-j(3\pi t + \pi/3)}) = -\frac{1}{2j} e^{-j\pi/3} e^{-j3\pi t} + \frac{1}{2j} e^{j\pi/3} e^{j3\pi t}.$$

By inspection in the complex exponential Fourier series representation we therefore have  $\omega_0 = 3\pi$  and  $c_{-1}^* = c_1 = \frac{1}{2j} e^{j\pi/3}$ , with all other coefficients  $c_k$  zero. Since  $j = e^{j\pi/2}$  this gives

$$c_1 = \frac{1}{2} e^{j\pi/3} e^{-j\pi/2} = \frac{1}{2} e^{-j\pi/6}$$

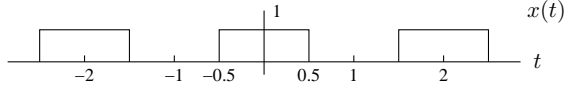
and the magnitude and phase plots are as follows:



(b) The trigonometric Fourier series requires  $\omega_0 = 3\pi$  and the coefficients satisfy  $a_0 = c_0 = 0$ ,  $a_1 = 2|c_1| = 1$  and  $b_1 = \angle c_1 = -\pi/6$ , with all other coefficients  $a_k$  and  $b_k$  zero.

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2. (5 marks) The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi} \sin(k\pi/2) & \text{otherwise.} \end{cases}$$

- (a) What is the total average power contained in the signal?  
 (b) How much signal power is contained in the frequency range  $|\omega| \geq 1.5\pi$ ?

Recall that Parseval's theorem states that

$$\frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

- (a) From the time domain representation the average signal power is

$$P = \frac{1}{2} \int_{-1}^1 x^2(t) dt = 1/2 \int_{-1/2}^{1/2} dt = 1/2 \text{ watts.}$$

The coefficient  $c_k$  corresponds to the component of the signal at frequency  $\omega = k\omega_0 = k\pi$  for integer  $k$ . Thus the total signal power in the range  $-1.5\pi < \omega < 1.5\pi$  is

$$P_{-1.5\pi < \omega < 1.5\pi} = \sum_{k=-1}^1 |c_k|^2 = (1/\pi)^2 + (1/2)^2 + (1/\pi)^2 = 0.4526 \text{ watts}$$

The power in the specified range must be the difference between the total power and this calculated power:

$$P_{|\omega| \geq 1.5\pi} = P - P_{-1.5\pi < \omega < 1.5\pi} = 0.5 - 0.4526 = 0.0474 \text{ watts,}$$

which is 9.48% of the total.

3. (5 marks) Plot the magnitude and phase spectrum (in other words  $|X(\omega)|$  and  $\angle X(\omega)$ ) of the signal  $x(t) = e^{-t}u(t)$ .

The Fourier transform of the given signal can be obtained from tables:

$$X(\omega) = \frac{1}{j\omega + 1}.$$

To generate the required plots this needs to be converted to an expression in polar form. Since  $\text{Re}(1 + j\omega) = 1$  and  $\text{Im}(1 + j\omega) = \omega$  the geometry of the complex plane implies that

$$1 + j\omega = \sqrt{1 + \omega^2} e^{j \arctan(\omega)}.$$

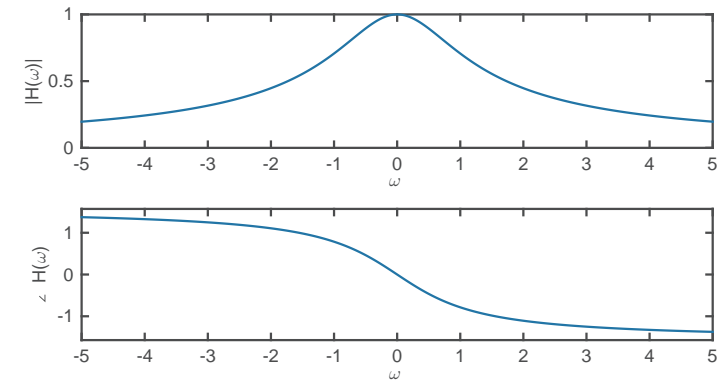
Thus

$$X(\omega) = \frac{1}{j\omega + 1} = \frac{1}{\sqrt{1 + \omega^2}} e^{-j \arctan(\omega)}$$

so

$$|X(\omega)| = \frac{1}{\sqrt{1 + \omega^2}} \quad \text{and} \quad \angle X(\omega) = -\arctan(\omega).$$

Plots are as follows:



4. (5 marks) Suppose  $y(t) = h(t) * e^{j\omega_0 t}$  with

$$h(t) = e^{-2t}u(t),$$

where  $*$  indicates the convolution operator.

(a) Find  $H(\omega)$ .

(b) Show that

$$Y(\omega) = \frac{2\pi}{j\omega_0 + 2} \delta(\omega - \omega_0).$$

(c) Find  $y(t)$ .

(a) The Fourier transform can be obtained from tables:

$$H(\omega) = \frac{1}{j\omega + 2}.$$

(b) Defining  $v(t) = e^{j\omega_0 t}$  we have  $y(t) = h(t) * v(t)$ . From the "Convolution in time domain" property

$$Y(\omega) = H(\omega)V(\omega).$$

However, from the Fourier pair

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0)$$

we know that  $V(\omega) = 2\pi\delta(\omega - \omega_0)$ . Thus

$$\begin{aligned} Y(\omega) &= H(\omega)V(\omega) = \frac{1}{j\omega + 2} 2\pi\delta(\omega - \omega_0) = \frac{2\pi}{j\omega + 2} \delta(\omega - \omega_0) \\ &= \frac{2\pi}{j\omega_0 + 2} \delta(\omega - \omega_0). \end{aligned}$$

The last equality follows from the sifting property since the impulse is at  $\omega = \omega_0$ .

(c) Applying the linearity property to the Fourier pair from the previous part gives the new pair

$$ae^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} a2\pi\delta(\omega - \omega_0),$$

which is true for all  $a$ . Letting  $a = \frac{1}{j\omega_0 + 2}$  yields the pair

$$\frac{1}{j\omega_0 + 2} e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} \frac{2\pi}{j\omega_0 + 2} \delta(\omega - \omega_0),$$

so the required time domain signal is

$$y(t) = \frac{1}{j\omega_0 + 2} e^{j\omega_0 t}.$$

## INFORMATION SHEET

### Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{ a }X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of $t$	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}  X(\omega) ^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

### Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$
1 $(-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad (c \text{ any real number})$
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad (\omega_0 \text{ any real number})$
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2 t }{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2 \omega }{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

### Trigonometric identities

$$\begin{aligned} \sin(-\theta) &= -\sin(\theta) & \cos(-\theta) &= \cos(\theta) & \tan(-\theta) &= -\tan(\theta) & \sin^2(\theta) + \cos^2(\theta) &= 1 \\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta) & \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ e^{j\theta} &= \cos(\theta) + j\sin(\theta) \end{aligned}$$