EEE2035F: Signals and Systems I

Class Test 2

25 April 2016

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Consider the signal

$$x(t) = \sin\left(3\pi t + \frac{\pi}{3}\right).$$

(a) Show that the signal can be written as a complex exponential Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(3\pi)t}$$

with $c_1 = c_{-1}^* = \frac{1}{2j}e^{j\pi/3}$ and all other coefficients zero. Plot the magnitude and phase of the coefficients c_k as a function of k.

(b) The signal $x(t)$ can also be expressed as a standard trigonometric Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t + b_k).$$

Specify ω_0 and the coefficients a_k and b_k.

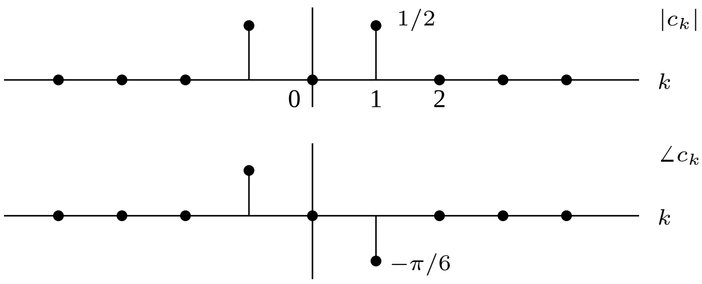
(a) Using Euler's expansion we get

$$x(t) = \frac{1}{2j}(e^{j(3\pi t+\pi/3} - e^{-j(3\pi t+\pi/3)}) = -\frac{1}{2j}e^{-j\pi/3}e^{-j3\pi t} + \frac{1}{2j}e^{j\pi/3}e^{j3\pi t}.$$

By inspection in the complex exponential Fourier series representation we therefore have $\omega_0 = 3\pi$ and $c_{-1}^* = c_1 = \frac{1}{2j}e^{j\pi/3}$, with all other coefficients c_k zero. Since $j = e^{j\pi/2}$ this gives

$$c_1 = \frac{1}{2}e^{j\pi/3}e^{-j\pi/2} = \frac{1}{2}e^{-j\pi/6}$$

and the magnitude and phase plots are as follows:



(b) The trigonometric Fourier series requires $\omega_0 = 3\pi$ and the coefficients satisfy $a_0 = c_0 = 0$, $a_1 = 2|c_1| = 1$ and $b_1 = \angle c_1 = -\pi/6$, with all other coefficients a_k and b_k zero.

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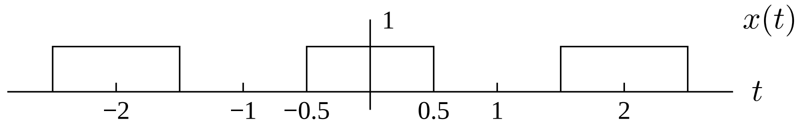
2. (5 marks) The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi}\sin(k\pi/2) & \text{otherwise.} \end{cases}$$

(a) What is the total average power contained in the signal?

(b) How much signal power is contained in the frequency range $|\omega| \geq 1.5\pi$?

Recall that Parseval's theorem states that

$$\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

(a) From the time domain representation the average signal power is

$$P = \frac{1}{2}\int_{-1}^{1} x^2(t)dt = 1/2\int_{-1/2}^{1/2} dt = 1/2 \text{ watts.}$$

The coefficient c_k corresponds to the component of the signal at frequency $\omega = k\omega_0 = k\pi$ for integer k. Thus the total signal power in the range $-1.5\pi < \omega < 1.5\pi$ is

$$P_{-1.5\pi<\omega<1.5\pi} = \sum_{k=-1}^{1} |c_k|^2 = (1/\pi)^2 + (1/2)^2 + (1/\pi)^2 = 0.4526 \text{ watts}$$

The power in the specified range must be the difference between the total power and this calculated power:

$$P_{|\omega|\geq 1.5\pi} = P - P_{-1.5\pi<\omega<1.5\pi} = 0.5 - 0.4526 = 0.0474 \text{ watts,}$$

which is 9.48% of the total.

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3. (5 marks) Plot the magnitude and phase spectrum (in other words $|X(\omega)|$ and $\angle X(\omega)$) of the signal $x(t) = e^{-t}u(t)$.

The Fourier transform of the given signal can be obtained from tables:

$$X(\omega) = \frac{1}{j\omega + 1}.$$

To generate the required plots this needs to be converted to an expression in polar form. Since $\text{Re}(1 + j\omega) = 1$ and $\text{Im}(1 + j\omega) = \omega$ the geometry of the complex plane implies that

$$1 + j\omega = \sqrt{1 + \omega^2} e^{j \arctan(\omega)}.$$

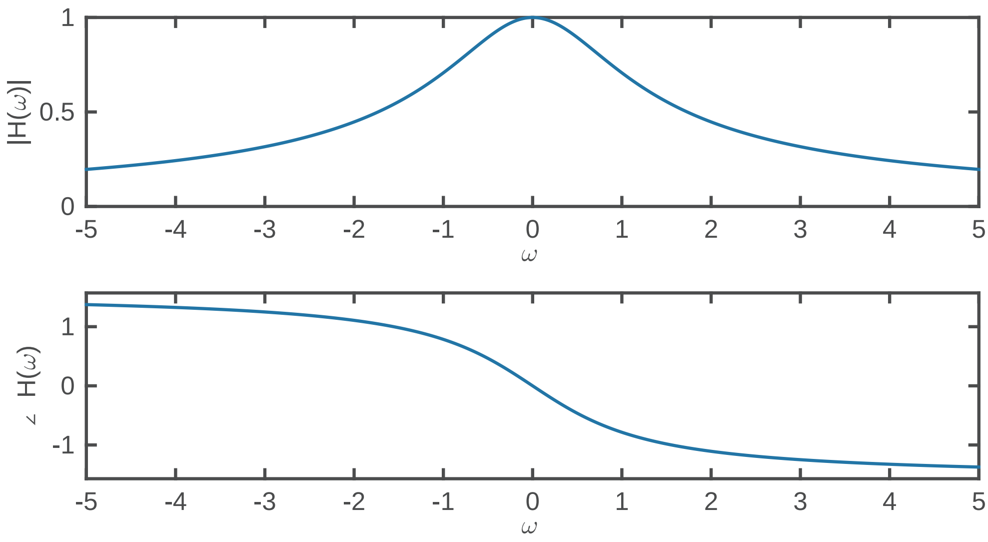
Thus

$$X(\omega) = \frac{1}{j\omega + 1} = \frac{1}{\sqrt{1 + \omega^2}} e^{-j \arctan(\omega)}$$

so

$$|X(\omega)| = \frac{1}{\sqrt{1 + \omega^2}} \quad \text{and} \quad \angle X(\omega) = -\arctan(\omega).$$

Plots are as follows:



4. (5 marks) Suppose $y(t) = h(t) * e^{j\omega_0 t}$ with

$$h(t) = e^{-2t}u(t),$$

where $*$ indicates the convolution operator.

(a) Find $H(\omega)$.

(b) Show that

$$Y(\omega) = \frac{2\pi}{j\omega_0 + 2}\delta(\omega - \omega_0).$$

(c) Find $y(t)$.

(a) The Fourier transform can be obtained from tables:

$$H(\omega) = \frac{1}{j\omega + 2}.$$

(b) Defining $v(t) = e^{j\omega_0 t}$ we have $y(t) = h(t) * v(t)$. From the "Convolution in time domain" property

$$Y(\omega) = H(\omega)V(\omega).$$

However, from the Fourier pair

$$e^{j\omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\delta(\omega - \omega_0)$$

we know that $V(\omega) = 2\pi\delta(\omega - \omega_0)$. Thus

$$\begin{aligned} Y(\omega) = H(\omega)V(\omega) &= \frac{1}{j\omega + 2}2\pi\delta(\omega - \omega_0) = \frac{2\pi}{j\omega + 2}\delta(\omega - \omega_0) \\ &= \frac{2\pi}{j\omega_0 + 2}\delta(\omega - \omega_0). \end{aligned}$$

The last equality follows form the sifting property since the impulse is at $\omega = \omega_0$.

(c) Applying the linearity property to the Fourier pair from the previous part gives the new pair

$$ae^{j\omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad a2\pi\delta(\omega - \omega_0),$$

which is true for all a. Letting $a = \frac{1}{j\omega_0 + 2}$ yields the pair

$$\frac{1}{j\omega_0 + 2}e^{j\omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{2\pi}{j\omega_0 + 2}\delta(\omega - \omega_0),$$

so the required time domain signal is

$$y(t) = \frac{1}{j\omega_0 + 2}e^{j\omega_0 t}.$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|X(\omega)\|^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\|t\|}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\|\omega\|}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$