EEE2035F: Signals and Systems I

Class Test 2

20 April 2015

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) The *step* response of a linear time-invariant system is given by

$$g(t) = [2 + 3e^{-4(t+1)}]u(t+1).$$

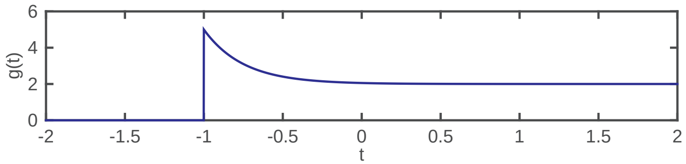
(a) Plot $g(t)$.

(b) Find the impulse response of the system.

(c) Is the system causal or non-causal? Why?

Note: the step response of a system is the output when the input is $u(t)$.

(a) Plot as follows:



(b) The impulse response is the derivative of the step response:

$$h(t) = 5\delta(t+1) - 12e^{-4(t+1)}u(t+1).$$

(c) Since the impulse response is not zero for all $t < 0$ the system is not causal.

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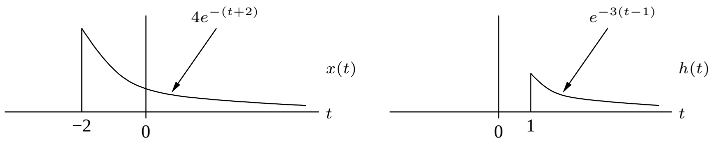
2. (5 marks) Given $x(t)$ and $h(t)$ shown below, find the convolution $y(t) = x(t) * h(t)$ for all values of t.



Center the functions to make the convolution easier: let

$$x_0(t) = x(t-2) = 4e^{-t}u(t) \qquad \text{and} \qquad h_0(t) = h(t+1) = e^{-3t}u(t)$$

and calculate $y_0(t) = x_0(t) * h_0(t)$. Using graphical methods we see that $y_0(t) = 0$ for $t < 0$. For $t \geq 0$ we have

$$\begin{aligned} y_0(t) &= \int_0^t x_0(s)h_0(t-s)ds = \int_0^t 4e^{-s}e^{-3(t-s)}ds = 4e^{-3t}\int_0^t e^{2s}ds \\ &= 2e^{-3t}(e^{2t}-1) = 2e^{-t} - 2e^{-3t}. \end{aligned}$$

Thus $y_0(t) = 2(e^{-t} - e^{-3t})u(t)$.

Finally applying the shifting property of convolution twice we have

$$y(t+1) = x(t) * h(t+1) = x(t) * h_0(t)$$

and

$$y(t+1-2) = x(t-2) * h_0(t) = x_0(t) * h_0(t) = y_0(t),$$

so $y(t-1) = y_0(t)$ or

$$y(t) = y_0(t+1) = 2(e^{-(t+1)} - e^{-3(t+1)})u(t+1).$$

Note that we could also find the required shift by considering graphical convolution of the original functions. It's easy to see then that the terms in the convolution integral begin to overlap for $t = -1$, so the result $y_0(t)$ must be shifted to start at this location.

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3. (5 marks) Let $x(t)$ be a real periodic function, with fundamental frequency $\omega_0 = 4$ and with Fourier series in exponential form:

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{j^k |k|}{1+k^2} e^{jk4t}$$

(a) Plot the magnitude and phase of the Fourier series coefficients over the range $k = -4$ to $k = 4$.

(b) Determine the Fourier series of the signal in trigonometric form.

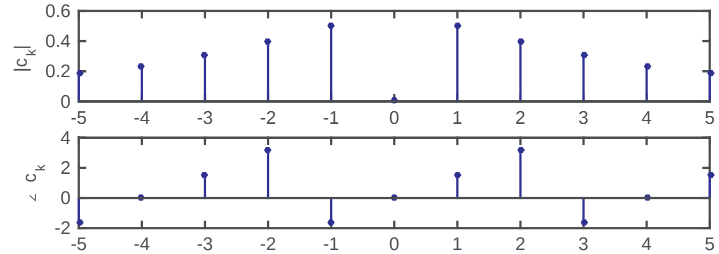
(c) How much power is contained in the second harmonic?

The coefficients can be written in polar form as

$$c_k = \frac{j^k|k|}{1+k^2} = \frac{|k|}{1+k^2}j^k = \frac{|k|}{1+k^2}e^{j\pi/2k},$$

so $|c_k| = |k|/(1+k^2)$ and $\angle c_k = \pi/2k$.

(a) Magnitude and phase plots as follows:



(b) Letting $c_k = |c_k|e^{j\angle c_k}$ and noting that $c_{-k} = c_k^*$ for a real signal we have $c_{-k} = |c_k|e^{-j\angle c_k}$. Each corresponding pair of terms in the sum can be written as

$$\begin{aligned} c_{-k}e^{-jk4t} + c_k e^{jk4t} &= |c_k|e^{-j\angle c_k}e^{-jk4t} + |c_k|e^{j\angle c_k}e^{jk4t} \\ &= 2|c_k|\frac{1}{2}(e^{-j(k4t+\angle c_k} + e^{j(k4t+\angle c_k}) = 2|c_k|\cos(k4t + \angle c_k), \end{aligned}$$

so

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2|c_k|\cos(k4t + \angle c_k) = \sum_{k=1}^{\infty} \frac{2|k|}{1+k^2}\cos(k4t + \pi/2k)$$

(since $c_0 = 0$).

(c) The power contained is $|c_{-2}|^2 + |c_2|^2 = 2(2/5)^2 = 8/25$ Watts.

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4. (5 marks) An LTI system has impulse response $h(t) = e^{-t}u(t)$.

(a) Use tables to find the Fourier transform $H(\omega)$ of $h(t)$.

(b) Use convolution to find the output $y(t)$ when the input is $x(t) = e^{j5t}$. Show that this can be written as $y(t) = H(5)e^{j5t}$.

(c) Suppose the input to the system is the signal

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk5t}.$$

Write down an expression for the output $y(t)$.

(a) The transform is

$$H(\omega) = \frac{1}{j\omega + 1}.$$

(b) For input $x(t) = e^{j5t}$ the output is

$$\begin{aligned} y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(s)x(t-s)ds = \int_{-\infty}^{\infty} e^{-s}u(s)e^{j5(t-s)}ds \\ &= \int_{0}^{\infty} e^{-(j5+1)s}e^{j5t}ds = -\frac{1}{j5+1}\left[e^{-(j5+1)s}\right]_{s=0}^{\infty} e^{j5t} \\ &= \frac{1}{j5+1}e^{j5t} = H(5)e^{j5t}. \end{aligned}$$

(c) In general the response to input $e^{j\omega t}$ is $H(\omega)e^{j\omega t}$. Applying this to each term in the given $x(t)$ yields the required result:

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(5k)e^{jk5t} = \sum_{k=-\infty}^{\infty} c_k \frac{1}{j5k+1}e^{jk5t}.$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \mathrm{sinc}\frac{\tau\omega}{2\pi}$
$\tau \mathrm{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right)p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\mathrm{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right)p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$