EEE2035F: Signals and Systems I

Class Test 2

22 April 2013

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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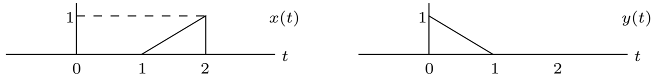
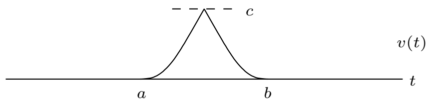
1. (5 marks) Consider the two signals below:

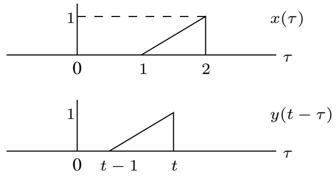


The convolution $v(t) = x(t) * y(t)$ is as follows:



Specify a, b, and c.

The picture for the graphical approach to convolution is as follows:



From this we see that the convolution is zero for $t < 1$ and for $t > 3$, so $a = 1$ and $b = 3$. The convolution is given by

$$v(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

with

$$x(t) = \begin{cases} -1+t & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the maximum occurs for $t = 2$. The corresponding value is

$$\begin{aligned} v(2) &= \int_{-\infty}^{\infty} x(\tau)y(2-\tau)d\tau = \int_{1}^{2} [1-(2-\tau)](-1+\tau)d\tau \\ &= \int_{1}^{2} (\tau-1)^2 d\tau = \int_{1}^{2} \tau^2 - 2\tau + 1 d\tau = \left[\frac{1}{3}\tau^3 - \tau^2 + \tau\right]_{\tau=1}^{2} \\ &= 8/3 - 4 + 2 - (1/3 - 1 + 1) = 1/3, \end{aligned}$$

so $c = 1/3$. Note that the algebra in the last part could be made much more simpler by thinking about convolution with $x(t)$ shifted to the right by 1.

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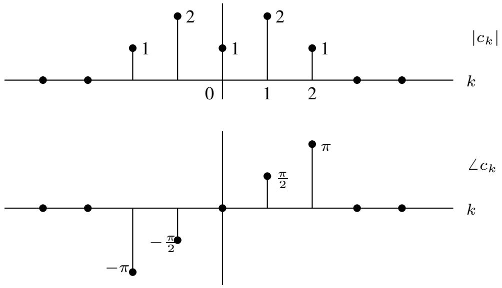
2. (5 marks) A signal $x(t)$ with fundamental frequency $\omega_0 = 2\pi$ radians/second has the Fourier series coefficients shown below:



(a) Is the signal real? Why?

(b) Write a time domain expression for the signal.

(c) What is the total power in the signal?

(a) We are assuming that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

For a real signal we must have $c_k = c_{-k}^*$, or $|c_k| = |c_{-k}|$ and $\angle c_k = -\angle c_{-k}$. Since we observe that the magnitude plot is even and the phase plot is odd this condition is met, so the signal is real.

(b) The signal can be written as

$$\begin{aligned} x(t) &= c_{-2}e^{-j4\pi t} + c_{-1}e^{-j2\pi t} + c_0 + c_1 e^{j2\pi t} + c_2 e^{j4\pi t} \\ &= e^{-j\pi}e^{-j4\pi t} + 2e^{-j\pi/2}e^{-j2\pi t} + 1 + 2e^{j\pi/2}e^{j2\pi t} + e^{j\pi}e^{j4\pi t} \\ &= e^{-j(4\pi t+\pi)} + 2e^{-j(2\pi t+\pi/2)} + 1 + 2e^{j(2\pi t+\pi/2)} + e^{j(4\pi t+\pi)} \\ &= 1 + 4\cos(2\pi t + \pi/2) + 2\cos(4\pi t + \pi). \end{aligned}$$

Writing the solution as a real function is probably not essential, although it does show that you know what you're doing.

(c) By Parseval's theorem the total average power is

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2 = 1 + 4 + 1 + 4 + 1 = 11 \text{ Watts}.$$

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3. (5 marks) The signal

$$x(t) = 2\cos(2t + \pi/3) + \sin(3t)$$

has a Fourier series representation of the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Specify ω_0, and find and plot the magnitude and phase of the Fourier series coefficients as a function of the frequency ω.

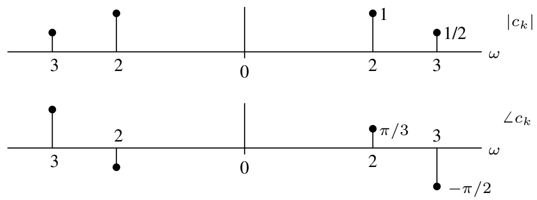
The signal can be written as

$$\begin{aligned} x(t) &= 2\frac{1}{2}(e^{j(2t+\pi/3)} + e^{-j(2t+\pi/3)}) + \frac{1}{2j}(e^{j3t} - e^{-j3t}) \\ &= -\frac{1}{2j}e^{-j3t} + e^{-j\pi/3}e^{-j2t} + e^{j\pi/3}e^{j2t} + \frac{1}{2j}e^{j3t} \end{aligned}$$

This can be expressed in the Fourier series form with $\omega_0 = 1$ radian/second,

$$c_{-3} = -\frac{1}{2j}, \quad c_{-2} = e^{-j\pi/3}, \quad c_2 = e^{j\pi/3}, \quad \text{and} \quad c_3 = \frac{1}{2j},$$

and all other coefficients zero. Since $c_3 = \frac{1}{2j} = -\frac{1}{2}j = \frac{1}{2}e^{-j\pi/2}$ and $c_k = c_{-k}^*$ the plots are as follows:



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4. (5 marks) Use the definition of the Fourier transform (that is, without using tables or properties) to compute the frequency domain representations of the following signals:
   (a) $x_1(t) = \delta(t-c)$.
   (b) $x_2(t) = e^{at}u(-t)$, where $a > 0$.

(a) The transform is

$$\mathcal{F}\{\delta(t-c)\} = \int_{-\infty}^{\infty} \delta(t-c)e^{-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} \delta(t-c)e^{-j\omega c}dt = e^{-j\omega c}\int_{-\infty}^{\infty} \delta(t-c)dt = e^{-j\omega c}.$$

(b) The transform is as follows:

$$\mathcal{F}\{e^{at}u(-t)\} = \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt$$
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}dt = \frac{1}{a-j\omega}\left[e^{at}e^{-j\omega t}\right]_{t\to-\infty}^{t=0}$$
$$= \frac{1}{a-j\omega}(1-0) = \frac{1}{a-j\omega}.$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t)+bv(t) \leftrightarrow aX(\omega)+bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a>0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n=1,2,\ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n=1,2,\ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Convolution in time domain	$x(t)*v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega)*V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5+u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c}$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b>0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$ (ω_0 any real number)
$p_\tau(t)$	$\tau \,\mathrm{sinc}\, \frac{\tau\omega}{2\pi}$
$\tau \,\mathrm{sinc}\, \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1-\frac{2\lvert t\rvert}{\tau}\right)p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\mathrm{sinc}^2\,\frac{\tau t}{4\pi}$	$2\pi\left(1-\frac{2\lvert\omega\rvert}{\tau}\right)p_\tau(\omega)$
$\cos(\omega_0 t+\theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0)+e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t+\theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0)-e^{j\theta}\delta(\omega-\omega_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{T})$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta) \qquad \sin^2(\theta)+\cos^2(\theta) = 1$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \qquad \cos(2\theta) = \cos^2(\theta)-\sin^2(\theta) = 2\cos^2(\theta)-1 = 1-2\sin^2(\theta)$

$\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta)+j\sin(\theta)$