EEE2035F: Signals and Systems I

Class Test 2

23 April 2012

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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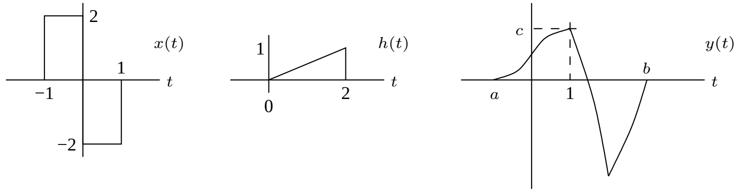
1. (5 marks) For the following signals, $y(t) = h(t) * x(t)$:

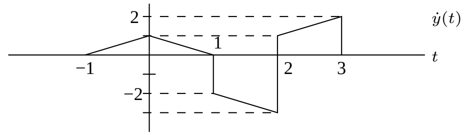


Find the values of a, b, and c, justifying all answers.

From the properties of convolution we have $\dot{y}(t) = h(t) * \dot{x}(t)$. However, $\dot{x}(t) = 2\delta(t-1) - 4\delta(t) + 2\delta(t-1)$, so

$$\dot{y}(t) = 2h(t+1) - 4h(t) + 2h(t-1).$$

A plot is below:



The signal given is the indefinite integral:

$$y(t) = \int_{-\infty}^{t} \dot{y}(\lambda) d\lambda.$$

The output will be zero prior to $t = -1$, at which point it will start to go positive. Thus $a = -1$. When $t = 1$ the total area to the left is 1, so $c = 1$. When $t = 3$ the total area to the left will be zero, and remains zero for $t > 3$. Thus $b = 3$.

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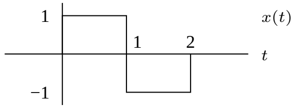
2. (5 marks) The impulse response of a LTI system is given by $h(t) = \frac{1}{\tau_0} e^{-t/\tau_0} u(t)$.

(a) Find the step response $g(t)$ of the system, or the response of the system to the unit step input $x(t) = u(t)$.

(b) Find a simple expression for the response $y(t)$ of the system to the input $x(t)$ below:



(Hint: the result from part (a) can be useful here. If you couldn't find the answer for that part, then just express the response in terms of $g(t)$.)

(a) If $u(t) \longrightarrow g(t)$ is a valid input-output pair then from the derivative property of convolution

$$\delta(t) = \frac{d}{dt}u(t) \quad \longrightarrow \quad \frac{d}{dt}g(t)$$

is a valid pair. But the left hand side is just the impulse response, so the right hand side must be the impulse response. Thus $h(t) = \dot{g}(t)$, or

$$g(t) = \int_{-\infty}^{t} h(\lambda) d\lambda.$$

When $t < 0$ we have $g(t) = 0$. For $t \geq 0$ the output is

$$\begin{aligned} g(t) &= \int_0^t \frac{1}{\tau_0} e^{-\frac{1}{\tau_0}\lambda} d\lambda = \frac{1}{\tau_0}\left[-\tau_0 e^{-\frac{1}{\tau_0}\lambda}\right]_{\lambda=0}^{t} \\ &= \frac{1}{\tau_0}\left(-\tau_0 e^{-\frac{1}{\tau_0}t} + \tau_0\right) = 1 - e^{-\frac{1}{\tau_0}t}. \end{aligned}$$

Thus we can write $g(t) = (1 - e^{-\frac{1}{\tau_0}t})u(t)$.

(b) The input can be written as $x(t) = u(t) - 2u(t-1) + u(t-2)$. However, by linearity and time invariance the following are all valid input-output pairs:

$$u(t) \longrightarrow g(t), \quad -2u(t-1) \longrightarrow -2g(t-1), \quad \text{and} \quad u(t-2) \longrightarrow g(t-2).$$

The output is therefore

$$\begin{aligned} y(t) &= g(t) - 2g(t-1) + g(t-2) \\ &= (1 - e^{-\frac{1}{\tau_0}t})u(t) - 2(1 - e^{-\frac{1}{\tau_0}(t-1)})u(t-1) + (1 - e^{-\frac{1}{\tau_0}(t-2)})u(t-2). \end{aligned}$$

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3. (5 marks) Consider the periodic signal $x(t)$ given by the expression

$$x(t) = (2+2j)e^{-j3t} - 3je^{-j2t} + 5 + 3je^{j2t} + (2-2j)e^{j3t}.$$

(a) What is the fundamental frequency of $x(t)$?

(b) Sketch the Fourier spectrum (the Fourier series coefficients).

(c) Is $x(t)$ a real signal? Why?

(d) What is the total power of $x(t)$?

(a) The Fourier series representation of the signal is

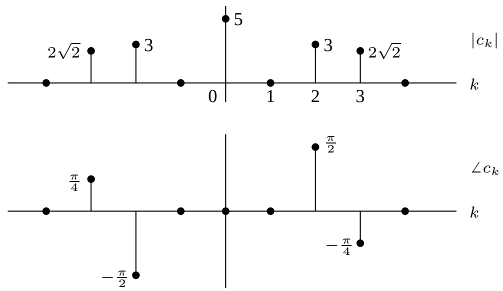
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

By inspection we observe that the fundamental frequency is $\omega_0 = 1$.

(b) The coefficients can also be found by inspection:

$$c_0 = 5, \quad c_2 = c_{-2}^* = 3j = 3e^{j\frac{\pi}{2}}, \quad \text{and} \quad c_3 = c_{-3}^* = 2 - 2j = 2\sqrt{2}e^{-j\frac{\pi}{4}}$$

with all other coefficients zero. These are plotted below:



(c) We know the signal is real because $c_k = c_{-k}^*$ for all k. (Magnitude is even and phase is odd.)

(d) From Parseval's theorem the total power is

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2 = (2\sqrt{2})^2 + 3^2 + 5^2 + 3^2 + (2\sqrt{2})^2 = 59 \text{ Watts.}$$

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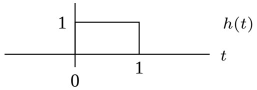
4. (5 marks) The impulse response of a LTI system is known to be



(a) Use the definition of convolution to show that when the input is $x(t) = e^{j\omega_0 t}$, the output is $y(t) = H(\omega_0)e^{j\omega_0 t}$ with

$$H(\omega) = \frac{1 - e^{-j\omega}}{j\omega}.$$

(b) Using this result (or otherwise) find the output when the input is $x(t) = e^{j2\pi t}$.

(c) Suppose now that the input is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

with $c_0 = 3$, $c_1 = c_{-1}^* = 2$ and $c_2 = c_{-2}^* = 1$, and all other coefficients zero. The output can be expressed as

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t},$$

Specify the values for d_k.

(a) The output is

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = \int_0^1 e^{j\omega_0(t-\lambda)}d\lambda = e^{j\omega_0 t}\left[\frac{-1}{j\omega_0}e^{-j\omega_0\lambda}\right]_{\lambda=0}^{1}$$
$$= \frac{1 - e^{-j\omega_0}}{j\omega_0}e^{j\omega_0 t} = H(\omega_0)e^{j\omega_0 t}.$$

(b) The output will be

$$y(t) = H(2\pi)e^{j2\pi t} = \frac{1 - e^{-j2\pi}}{j2\pi}e^{j2\pi t} = 0.$$

(c) If the action of the system is denoted by the transformation $T\{\cdot\}$, then by linearity and time invariance we have

$$y(t) = T\{x(t)\} = T\{x(t)\} = T\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} T\{c_k e^{jk\omega_0 t}\}$$
$$= \sum_{k=-\infty}^{\infty} c_k T\{e^{jk\omega_0 t}\} = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0)e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}.$$

Therefore

$$d_k = c_k\frac{1 - e^{-jk\omega_0}}{jk\omega_0} = c_k\frac{1 - e^{-jk\pi}}{jk\pi}.$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$
$\tau \operatorname{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\operatorname{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$