EEE2035F: Signals and Systems I

Class Test 2

21 April 2011

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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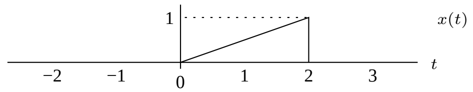
1. (5 marks) suppose $x(t)$ is the signal below



and let

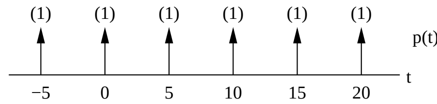
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

be an impulse train with $T = 5$.

(a) Plot $p(t)$ in the time domain.

(b) Find and plot $y(t) = p(t) * x(t)$.

(a) Plot as follows:

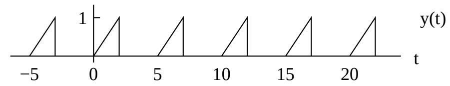


(b) The signal is

$$y(t) = \left(\sum_{n=-\infty}^{\infty} \delta(t - nT) \right) * x(t) = \sum_{n=-\infty}^{\infty} [\delta(t - nT) * x(t)]$$

$$= \sum_{n=-\infty}^{\infty} x(t - nT) = \sum_{n=-\infty}^{\infty} x(t - 5n).$$

Plot as follows:



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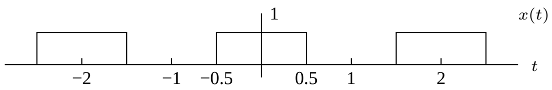
2. (5 marks) The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi}\sin(k\pi/2) & \text{otherwise.} \end{cases}$$

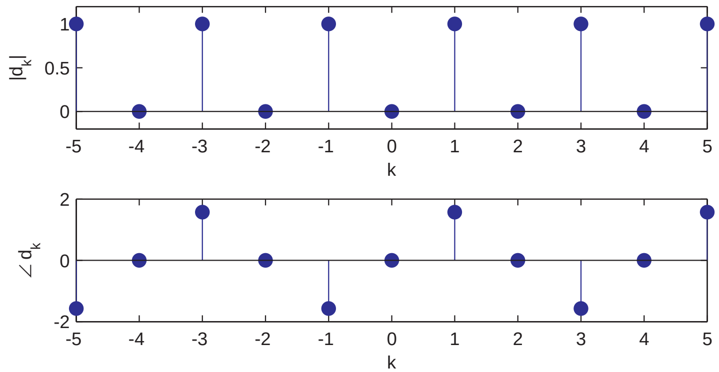
Use this information to find a Fourier series expansion for the signal $y(t) = \frac{d}{dt}x(t)$, and plot the magnitude and phase spectrum of $y(t)$ over the range $k = -5$ to $k = 5$.

We can write

$$y(t) = \frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} c_k \frac{d}{dt} e^{jk\pi t}$$

$$= \sum_{k=-\infty}^{\infty} c_k jk\pi e^{jk\pi t}.$$

This is in the form of a Fourier series $y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$ with $d_k = jk\pi c_k$, the values of which can be obtained from the expressions given. Plots as follows:



3. (5 marks) An LTI system is driven by the input signal $x(t) = e^{j\frac{\pi}{2}t}$.

(a) Find the output $y_1(t)$ if the impulse response of the system is $h_1(t) = \delta(t-1)$.

(b) Find the output $y_2(t)$ if the impulse response of the system is $h_2(t) = p_1(t)$ (the pulse of unit width centered on the origin).

(a) The system delays the input by one time unit, so the output will be

$$y_1(t) = x_1(t-1) = e^{j\frac{\pi}{2}(t-1)}.$$

(b) The output will be

$$y_2(t) = \int_{-\infty}^{\infty} h_2(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} p_1(\tau)e^{j\frac{\pi}{2}(t-\tau)}d\tau$$

$$= \left(\int_{-\infty}^{\infty} p_1(\tau)e^{-j\frac{\pi}{2}\tau}d\tau\right) e^{j\frac{\pi}{2}t} = P_1(\pi/2)e^{j\frac{\pi}{2}t},$$

where $P_1(\omega) = \mathcal{F}\{p_1(t)\}$. But from Fourier tables (or otherwise) we know that

$$P_1(\omega) = \text{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{2}{\omega}\sin(\omega/2),$$

so the output is

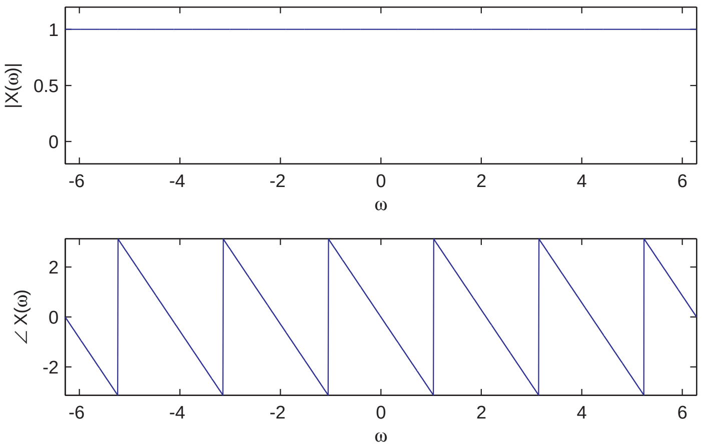
$$y_2(t) = P_1(\pi/2)e^{j\frac{\pi}{2}t} = \frac{4}{\sqrt{2}\pi}e^{j\frac{\pi}{2}t}.$$

4. (5 marks) Find the Fourier transform of the signal $x(t) = \delta(t-3)$ and plot the resulting magnitude and phase as a function of ω. The values in your phase plot should range between $-\pi$ and $-\pi$.

The Fourier transform is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t-3)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t-3)e^{-j\omega 3}dt$$
$$= e^{-j\omega 3}\int_{-\infty}^{\infty} \delta(t-3)dt = e^{-j\omega 3}.$$

This is already in polar form so $|X(\omega)| = 1$ and $\angle X(\omega) = -3\omega$. Plots as follows:



INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$