EEE2035F: Signals and Systems I

Class Test 2

16 April 2010

SOLUTIONS

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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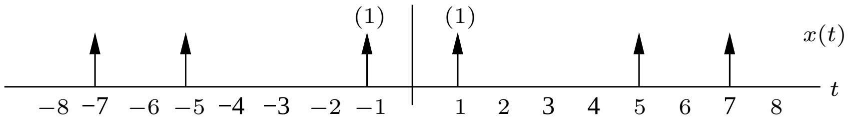
1. (5 marks) Find the complex exponential Fourier series representation of the periodic signal below:



Signal has period $T = 6$ so $\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$ rad/s in an expansion

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The Fourier coefficients are obtained by integrating over one period:

$$\begin{aligned} c_k &= \frac{1}{6}\int_{-6/2}^{6/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{6}\int_{-3}^{3}[\delta(t+1)+\delta(t-1)]e^{-jk\omega_0 t}dt \\ &= \frac{1}{6}\int_{-3}^{3}\delta(t+1)e^{-jk\omega_0 t}dt + \frac{1}{6}\int_{-3}^{3}\delta(t-1)e^{-jk\omega_0 t}dt \\ &= \frac{1}{6}e^{-jk\omega_0(-1)}\int_{-3}^{3}\delta(t+1)dt + \frac{1}{6}e^{-jk\omega_0(1)}\int_{-3}^{3}\delta(t-1)dt \\ &= \frac{1}{6}e^{jk\omega_0} + \frac{1}{6}e^{-jk\omega_0}, \end{aligned}$$

This completely specifies the coefficients, although it would be easier to simplify to

$$c_k = \frac{2}{6}\frac{1}{2}(e^{jk\omega_0} + e^{-jk\omega_0}) = \frac{1}{3}\cos(k\omega_0)$$

if you had to calculate them.

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2. (5 marks) Use transform properties to find the Fourier transform of

$$x(t) = e^{j5t} e^{-2(t-2)} u(t-2).$$

This can be decomposed as a time shift followed by multiplication by e^{j5t}. Start with

$$e^{-2t} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j\omega}$$

and apply time-shifting to get

$$e^{-2(t-2)} u(t-2) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j\omega} e^{-j\omega 2}$$

A frequency shift of 5 rad/s then yields the pair

$$e^{j5t} e^{-2(t-2)} u(t-2) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j(\omega-5)} e^{-j2(\omega-5)},$$

so the required Fourier transform is

$$X(\omega) = \frac{1}{2+j(\omega-5)} e^{-j2(\omega-5)}.$$

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3. (5 marks) Consider a signal with the following frequency domain representation:

$$X(\omega) = 2\pi[\delta(\omega - 5) + \delta(\omega + 5)].$$

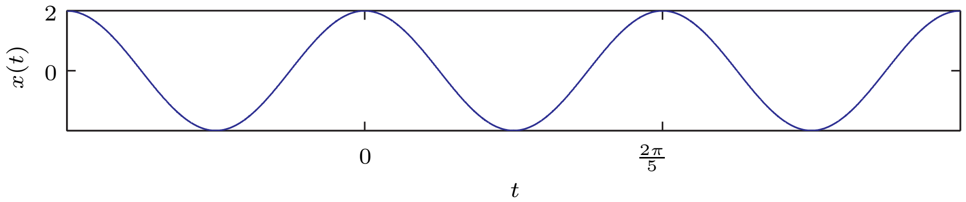
(a) Use the Fourier synthesis integral directly to show that $x(t) = 2\cos(5t)$.

(b) Plot $x(t)$ and $X(\omega)$.

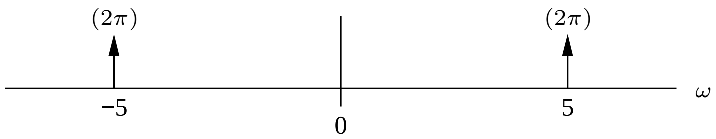
(a) The transform is

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi[\delta(\omega - 5) + \delta(\omega + 5)]e^{j\omega t}dt \\ &= e^{j5t}\int_{-\infty}^{\infty} \delta(\omega - 5)d\omega + e^{-j5t}\int_{-\infty}^{\infty} \delta(\omega - 5)d\omega \\ &= \frac{2}{2}(e^{j5t} + e^{-j5t}) = 2\cos(5t). \end{aligned}$$

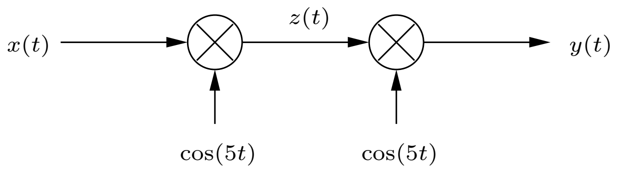
(b) Plots are



and



4. (5 marks) Consider the system shown below:



If the Fourier transform of $x(t)$ is $X(\omega) = p_1(\omega)$, then sketch the Fourier transform of $Y(\omega)$.

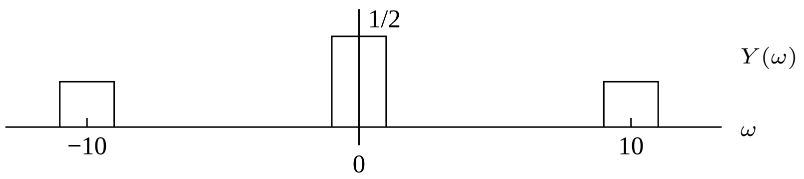
The transform of $z(t)$ is

$$Z(\omega) = \frac{1}{2}(X(\omega + 5) + X(\omega - 5)),$$

so

$$\begin{aligned} Y(\omega) &= \frac{1}{2}(Z(\omega + 5) + Z(\omega - 5)) \\ &= \frac{1}{4}[X((\omega + 5) + 5) + X((\omega + 5) - 5)] + \frac{1}{4}[X((\omega - 5) + 5) + X((\omega - 5) - 5)] \\ &= \frac{1}{4}X(\omega + 10) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega - 10). \end{aligned}$$

For $X(\omega) = p_1(\omega)$ this looks like



INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Frequency shift	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau\text{sinc}\frac{\tau\omega}{2\pi}$
$\tau\text{sinc}\frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\text{sinc}^2\frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$